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# STRUCTURAL SIMILITUDE AND DESIGN OF SCALED DOWN LAMINATED MODELS

by

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## Abstract

The excellent mechanical properties of laminated composite structures make them prime candidates for wide variety of applications in aerospace, mechanical and other branches of engineering. The enormous design flexibility of advanced composites is obtained at the cost of large number of design parameters. Due to complexity of the systems and lack of complete design based informations, designers tend to be conservative in their design. Furthermore, any new design is extensively evaluated experimentally until it achieves the necessary reliability, performance and safety. However, the experimental evaluation of composite structures are costly and time consuming. Consequently, it is extremely useful if a full-scale structure can be replaced by a similar scaled-down model which is much easier to work with. Furthermore, a dramatic reduction in cost and time can be achieved, if available experimental data of a specific structure can be used to predict the behavior of a group of similar systems.

This study investigates problems associated with the design of scaled models. Such study is important since it provides the necessary scaling laws, and the factors which affect the accuracy of the scale models.

Similitude theory is employed to develop the necessary similarity conditions (scaling laws). Scaling laws provide relationship between a full-scale structure and its scale model, and can be used to extrapolate the experimental data of a small, inexpensive, and testable model into design information for a large prototype. Due to large number of design parameters, the identification of the principal scaling laws by conventional method (dimensional analysis) is tedious. Similitude theory based on governing equations of the structural system is more direct and simpler in execution. The difficulty of making completely similar scale models often leads to accept certain type of distortion from exact duplication of the prototype (partial similarity). Both complete and partial similarity are discussed. The procedure consists of systematically observing the effect of each parameter and corresponding scaling laws. Then acceptable intervals and limitations for these parameters and scaling laws are discussed. In each case, a set of valid scaling factors and corresponding response scaling laws that accurately predict the response of prototypes from experimental models is introduced. The examples used include rectangular laminated plates under destabilizing loads, applied individually, vibrational characteristics of same plates, as well as cylindrical bending of beam-plates.

## DISCUSSION

The importance of employing small scale models in designing advanced composite structures has been gaining momentum in recent years. With a view to better understanding the applicability of these models in designing laminated composite structures, an analytical investigation was undertaken to assess the feasibility of their use. Employment of similitude theory to establish similarity among structural systems can save considerable expense and time, provided the proper scaling laws are found and validated.

Before small scale models can be used, the technical barriers that must be overcome are:

- What are the proper scale factors.
- What is the effect of these scale factors(scale effect).

In this study the limitation and acceptable interval of all parameters and corresponding scale factors are investigated. In the present studies, the material behavior was assumed to be linearly elastic. Therefore, scale effects are not present.

### Completed Tasks

An analytical investigation has been conducted in order to establish the applicability of similitude theory to laminated rectangular plates. Particular emphasis is placed on the case of free vibration and buckling of plates under uniaxial compressive and shear loads. Angle ply, cross ply and quasi-isotropic configurations were chosen for investigation.

### Current Tasks

#### i) *Nonlinear Kinematics, Linear Constitutive Relations*

The large deflection analysis is performed on composite beam- columns subjected to an eccentric axial compressive load. The loads are static. The objective of this investi-

gation is to develop and validate the scaling laws for nonlinear kinematic behavior (large deformation) of simple generic structural elements.

## ii) *Failure Analysis*

The laminate failure analysis is investigated for first ply failure of composite beam-columns. The beam-columns are subjected to static eccentric compressive loads. The objective of this study is to develop and validate the necessary scaling laws corresponding to stress analysis of laminated structures. In particular we like to predict the stress profile in prototype by projecting the corresponding stresses of its scale model.

## iii) *Curved Configuration*

The development of scaling laws which pertain to the elastic stability response of laminated shells is currently being investigated. Particular emphasis is placed on the case of buckling of orthotropic laminated cylindrical shells under axial compressive load.

## *Future Tasks*

The following research and development must be done before small scale model can be used in design and analysis of the composite laminated structures.

- Complete the investigation of curved configurations.
- Study the effect of boundary conditions.
- Study the applicability of scaled down models to geometrically stiffened structures.
- Study the effect of geometric imperfections.
- Design prototype and scaled down models for experimental validation ( at this time it is anticipated that tests will be performed at the "NASA Langley Research Center" labs).

## APPENDICES

- Structural Similitude For Laminated Structures.
- Structural Similitude and Scaling Laws for Cross-Ply Laminated Plates.
- Design of Scaled Down Models for Stability and Vibration Studies.

## STRUCTURAL SIMILITUDE FOR LAMINATED STRUCTURES

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**Abstract**—Due to special characteristics of advanced reinforced composite materials, they require extensive experimental evaluation. Thus, it is extremely useful to use available experimental data of specific structural systems to predict the behavior of all similar systems. This study describes the establishment of similarity conditions between two structural systems. Similarity conditions provide the relationship between a scale model and its prototype, and can be used to predict the behavior of the prototype by extrapolating the experimental data of the corresponding small scale model. Since satisfying all the similarity conditions simultaneously is in most cases impractical, distorted models with partial similarity (with at least one similarity condition relaxed) are employed. Establishing similarity conditions, based on direct use of governing equations, is discussed and the possibility of designing distorted models is investigated. The method is demonstrated through analysis of the cylindrical bending of orthotropic laminated beamplates subjected to transverse loads and buckling of symmetric laminated cross-ply rectangular plates subjected to uniaxial compression.

### NOMENCLATURE

$a$	plate length
$A_{ij}$	laminate extensional stiffnesses
$b$	plate width
$B_{ij}$	laminate coupling stiffnesses
$D$	plate flexural stiffness
$D_{ij}$	laminate flexural stiffnesses
$E_{ij}$	Young's moduli of elasticity
$F$	stiffness ratio
$h$	total laminate thickness
$k_{xx}$	bending curvature in the laminate
$m, n$	number of half waves in $x$ and $y$
$M$	cross-ply ratio
$M_{xx}$	moment resultant
$N$	number of layers
$N_{xx}$	stress resultant
$\bar{N}_x$	inplane load per unit width
$P$	total transverse load
$q$	transverse load intensity
$Q_{ij}, \bar{Q}_{ij}$	lamina stiffness elements
$t$	ply thickness
$u, v, w$	reference (midplane) surface displacements
$x, y, z$	reference axes
$\epsilon_{xx}^0$	midplane extensional strain
$\theta$	fiber orientation angle
$\Lambda$	transformation matrix
$\lambda_i$	scale factors
$\nu, \nu_{ij}$	Poisson's ratios
$\sigma_{xx}^{(k)}$	normal stress in the $k$ th lamina
$m$	model
$p$	prototype
pr.	predicted
th.	theoretical

### INTRODUCTION

Aircraft and spacecraft comprise the class of aerospace structures that require efficiency and wisdom in design, sophistication and accuracy in analysis and numerous and careful experimental evaluations of components and prototype, in order to achieve the necessary system reliability, performance and safety.



Preliminary and/or concept design entails the assemblage of system mission requirements, system expected performance and identification of components and their connections as well as of manufacturing and system assembly techniques. This is accomplished through experience based on previous similar designs, and through the possible use of models to simulate the entire system characteristics.

Detail design is heavily dependent on information and concepts derived from the previous step. This information identifies critical design areas which need sophisticated analyses, and design and redesign procedures to achieve the expected component performance. This step may require several independent *analysis* models, which, in many instances, require component testing.

The last step in the design process, before going to production, is the verification of the design. This step necessitates the production of large components and prototypes in order to test component and system analytical predictions and verify strength and performance requirements under the worst loading conditions that the system is expected to encounter in service.

Clearly then, full-scale testing is in many cases necessary and always very expensive. In the aircraft industry, in addition to full-scale tests, certification and safety necessitate large component static and dynamic testing. The C-141A ultimate static tests include eight wing tests, 17 fuselage tests and seven empennage tests (McDougal, 1987). Such tests are extremely difficult, time consuming and definitely absolutely necessary. Clearly, one should not expect that prototype testing will be totally eliminated in the aircraft industry. It is hoped, though, that we can reduce full-scale testing to a minimum.

Moreover, crashworthiness aircraft testing requires full-scale tests and several drop tests of large components. The variables and uncertainties in crash behavior are so many that the information extracted from each test, although extremely valuable, is nevertheless small by comparison to the expense. Moreover, each test provides enough new and unexpected phenomena, to require new tests, specially designed to explain the new observations.

In the building construction industry, when the skeleton frames are erected at the site, a specified sequence of erection events must be followed in order to avoid collapse. This was discovered through (expensive) experience, but it is not widely known. A small-scale testing of similar structures would definitely have been safer and less costly.

Finally, full-scale large component testing is necessary in other industries as well. Ship building, automobile and railway car construction all rely heavily on testing.

Regardless of the application, a scaled-down (by a large factor) model (*scale model*) which closely represents the structural behavior of the full-scale system (*prototype*) can prove to be an extremely beneficial tool. This possible development must be based on the existence of certain structural parameters that control the behavior of a structural system when acted upon by static and/or dynamic loads. If such structural parameters exist, a scaled-down replica can be built, which will duplicate the response of the full-scale system. The two systems are then said to be structurally similar. The term, then, that best describes this similarity is *structural similitude*.

Similarity of systems requires that the relevant system parameters be identical and these systems be governed by a unique set of characteristic equations. Thus, if a relation or equation of variables is written for a system, it is valid for all systems which are similar to it (Kline, 1965). Each variable in a model is proportional to the corresponding variable of the prototype. This ratio, which plays an essential role in predicting the relationship between the model and its prototype, is called the *scale factor*. In establishing similarity conditions between the model and prototype two procedures can be used, dimensional analysis and direct use of governing equations.

Models, as a design aid, have been used for many years, but the use of scientific models which are based on dimensional analysis was first discussed in a paper by Rayleigh (1915). Similarity conditions based on dimensional analysis have been used since Rayleigh's time (Macagno, 1971), but the applicability of the theory of similitude to structural systems was first discussed by Goodier and Thomson (1944) and later by Goodier (1950). They presented a systematic procedure for establishing similarity conditions based on dimensional analysis.

In the 1950s and 1960s many interesting works were published in this area. Most of these authors discussed similitude theory based on dimensional analysis. Kline (1965) gives a perspective of the method based on both dimensional analysis and the direct use of the governing equations. Szucs (1980) is particularly thorough on the topic of similitude theory. He explains the method with emphasis on the direct use of the governing equations of the system.

Due to special characteristics of advanced reinforced composite materials, they have been used extensively in weight efficient aerospace structures. Since reinforced composite components require extensive experimental evaluation, there is a growing interest in small scale model testing. Morton (1988) discusses the application of scaling laws for impact-loaded carbon-fiber composite beams. His work is based on dimensional analysis. Qian *et al.* (1990) conducted experimental studies of impact loaded composite plates, where the similarity conditions were obtained by considering the governing equations of the system. These works and many other experimental investigations have been conducted to characterize the size effect in material behavior for inelastic analysis.

In recent years, due to large dimensions and unique structural design of the proposed space station, small scale model testing and similitude analysis have been considered as the only option in order to gain experimental data. Shih *et al.* (1987), Letchworth *et al.* (1988), Hsu *et al.* (1989), and McGowan *et al.* (1990) discussed the possibility of scale model testing of space station geometries especially for vibration analysis. Most of these studies have used complete similarity between model and prototype.

The present study presents the applicability of small scale models, especially distorted models, in analysing the elastic behavior of large and complex structural systems. By applying similitude theory, we try to find a set of conditions between two similar structural systems (scaling laws). Later, these conditions can be used to design a model, the experimental data of which can be projected in order to predict the behavior of the prototype. The objectives of the investigation described herein are:

- create necessary similarity conditions in order to design an accurate distorted model;
- evaluate the derived similarity conditions analytically.

Similarity conditions provide the relationship between model and its prototype, and can be used to extrapolate the experimental data of a small and less expensive model in order to predict the behavior of the prototype. In all of our work in this area we will restrict ourselves to linearly elastic material behavior. Furthermore, it is assumed that the laminates are free of damage (delaminations, matrix cracking, fiber breaks, etc.).

#### THEORY OF SIMILITUDE

Similitude theory is concerned with establishing necessary and sufficient conditions of similarity between two phenomena. Establishing similarity between systems helps to predict the behavior of a system from the results of investigating other systems which have already been investigated or can be investigated more easily than the original system. Similitude among systems means similarity in behavior in some specific aspects. In other words, knowing how a given system responds to a specific input, the response of all similar systems to similar input can be predicted.

The behavior of a physical system depends on many parameters, i.e. geometry, material behavior, dynamic response and energy characteristic of the system. The nature of any system can be modeled mathematically in terms of its variables and parameters. A prototype and its scale model are two different systems with similar but not necessarily identical parameters. The necessary and sufficient conditions of similitude between prototype and its scale model require that the mathematical model of the scale model can be transformed to that of the prototype by a bi-unique mapping or vice versa (Szucs, 1980). It means, if vectors  $\mathbf{X}_p$  and  $\mathbf{X}_m$  are the characteristic vectors of the prototype and model, then we can find a transformation matrix  $\Lambda$  such that:

$$\mathbf{X}_p = \Lambda \mathbf{X}_m \quad \text{or} \quad \mathbf{X}_m = \Lambda^{-1} \mathbf{X}_p. \quad (1)$$

The elements of vector  $\mathbf{X}$  are all the parameters and variables of the system. A diagonal form of the transformation matrix  $\Lambda$  is the simplest form of transformation. The diagonal elements of the matrix are the scale factors of the pertinent elements of the characteristic vector  $\mathbf{X}$ .

$$\Lambda = \begin{bmatrix} \lambda_{x1} & 0 & \cdots & 0 \\ 0 & \lambda_{x2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{xn} \end{bmatrix}, \quad (2)$$

where  $\lambda_{x_i} = x_{ip}/x_{im}$  denotes the scale factor of  $x_i$ . In general the transformation matrix is not diagonal.

Since similitude theory gives many alternative ways for investigating a system, it has been used in areas which primarily involved many experimental investigations, such as fluid mechanics, aerodynamics, hydraulics and modal analysis.

In establishing similarity conditions between the model and prototype two procedures can be used, dimensional analysis and direct use of governing equations. The similarity conditions can be established either directly from the field equations of the system or, if it is a new phenomenon and the mathematical model of the system is not available, through dimensional analysis. In the second case, all of the variables and parameters which affect the behavior of the system must be known. By using dimensional analysis, an incomplete form of the characteristic equation of the system can be formulated. This equation is in terms of nondimensional products of variables and parameters of the system. Then, similarity conditions can be established on the basis of this equation.

In this study, we consider only direct use of the governing equations procedure. This method is more convenient than dimensional analysis, since the resulting similarity conditions are more specific. When governing equations of the system are used for establishing similarity conditions, the relationships among variables are forced by the governing equations of the system.

The field equations of a system with proper boundary and initial conditions characterize the behavior of the system in terms of its variables and parameters. If the field equations of the scale model and its prototype are invariant under transformation  $\Lambda$  and  $\Lambda^{-1}$ , then the two systems are completely similar [eqn (1)]. This transformation defines the scaling laws (similarity conditions) among all parameters, structural geometry and cause and response of the two systems.

In order to demonstrate the applicability of the method, we consider the following example. Suppose we want to design a reasonable (able to test) model for a large rectangular plate. The plate is simply supported at all edges and loaded with a uniform transverse load of intensity  $q$ . Assuming uniform cross section and isotropic material, the governing differential equations and boundary conditions are well known (Timoshenko and Woinowsky-Krieger, 1959):

$$\frac{d^4 w}{dx^4} + 2 \frac{d^4 w}{dx^2 dy^2} + \frac{d^4 w}{dy^4} = \frac{q}{D}, \quad (3)$$

and B.C. at  $x = 0, a$

$$w = 0$$

$$\frac{d^2 w}{dx^2} = 0, \quad (4)$$

and at  $y = 0, b$

$$w = 0$$

$$\frac{d^2 w}{dy^2} = 0. \quad (5)$$

For model and prototype we may write:

$$\frac{d^4 w_m}{dx_m^4} + 2 \frac{d^4 w_m}{dx_m^2 dy_m^2} + \frac{d^4 w_m}{dy_m^4} = \frac{q_m}{D_m}, \quad (6)$$

$$\frac{d^4 w_p}{dx_p^4} + 2 \frac{d^4 w_p}{dx_p^2 dy_p^2} + \frac{d^4 w_p}{dy_p^4} = \frac{q_p}{D_p}, \quad (7)$$

where subscripts  $m$  and  $p$  refer to model and prototype, respectively.

By defining scale factors  $\lambda_i$ , the variables of the prototype can be written as  $x_{ip} = \lambda_{xi} x_{im}$ . The response similarity conditions between model and prototype (complete similarity) are determined by substitution of the  $\lambda_{xi} x_{im}$  into the differential equation of the prototype and by requiring that the result be the differential equation of the model [eqn (5)].

$$\left(\frac{\lambda_w}{\lambda_x^4}\right) \frac{d^4 w_m}{dx_m^4} + 2 \left(\frac{\lambda_w}{\lambda_x^2 \lambda_y^2}\right) \frac{d^4 w_m}{dx_m^2 dy_m^2} + \left(\frac{\lambda_w}{\lambda_y^4}\right) \frac{d^4 w_m}{dy_m^4} = \left(\frac{\lambda_q}{\lambda_D}\right) \frac{q_m}{D_m}. \quad (8)$$

Equations (5) and (8) are the same if the terms in parentheses of eqn (8) are all equal.

$$\frac{\lambda_w}{\lambda_x^4} = \frac{\lambda_w}{\lambda_x^2 \lambda_y^2} = \frac{\lambda_w}{\lambda_y^4} = \frac{\lambda_q}{\lambda_D}. \quad (9)$$

Now to find the scaling laws from eqn (9), we have three choices. Dividing eqn (9) by first term, yields:

$$\lambda_x = \lambda_y, \quad \lambda_w = \frac{\lambda_q \lambda_x^4}{\lambda_D}. \quad (10)$$

Dividing eqn (9) by the second term, yields:

$$\lambda_x = \lambda_y, \quad \lambda_w = \frac{\lambda_q \lambda_x^2 \lambda_y^2}{\lambda_D}, \quad (11)$$

and finally dividing eqn (9) by third term:

$$\lambda_x = \lambda_y, \quad \lambda_w = \frac{\lambda_q \lambda_y^4}{\lambda_D}. \quad (12)$$

Note that all three, eqns (10)-(12), are equivalent. This means that as long as  $\lambda_x = \lambda_y$ , the behavioral condition that relates the response factor,  $\lambda_w$ , to the cause scale factor,  $\lambda_q$ , is the same for all three cases (complete similarity).

By applying similitude theory to a specific system, the result will be a set of conditions among pertinent parameters (scale factors of the parameters) of this system and its similar models. If all similarity conditions are satisfied, the two systems are completely similar. Suppose the system has  $m$  variables and similitude analysis of the governing equations of the system defines  $n$  relationships among  $m$  unknowns (scale factors of these variables). If the two systems are completely similar  $m - n$  scale factors can be chosen freely and the values of the other scale factors are found by using the  $n$  similarity conditions. The arbitrary scale factors are usually chosen based on the experimental facility, available material, and measurement techniques. By having the parameters of the prototype and scale factors, the model parameters can be calculated easily. Often complete similarity is difficult to achieve or even undesirable. This problem is usually caused by limitations on conducting the experiment. When at least one of the similarity conditions cannot be satisfied, partial similarity is achieved. In this case, the model which has some relaxation in similarity conditions is called a *distorted model*. Distorted models are more practical, since relaxation of each similarity condition eliminates some restrictions on the model design. These relaxations in the relationship between two systems cause model behavior to be different from that of the prototype. Understanding of these relaxations (and their effect on model behavior) can be used to modify the model test data so as to predict the behavior of the prototype. Since each

variable has different influence on the response of the system, the resulting similarity conditions have different influence. By understanding the effect of variables and similarity conditions over desired intervals, the similarity conditions which have the least influence can be neglected without introducing significant error (Kline, 1965).

#### APPLICATIONS

In this section, as an initial effort, similarity conditions are developed in order to design reasonable, distorted, scale models for orthotropic laminated beamplates and plates.

##### a. Cylindrical bending of laminated beamplates

a.1. *Deflections.* We desire to find the maximum deflection of beamplates. Beamplates are subjected to transverse line loads. By assuming that the displacement functions are independent of  $y$ , or  $u = u(x)$ ,  $v = 0$ ,  $w = w(x)$  (cylindrical bending), from Ashton and Whitney (1970) the governing differential equations and boundary conditions are reduced to:

$$\frac{d^4 w}{dx^4} = \frac{qA_{11}}{A_{11}D_{11} - B_{11}^2}, \quad (13)$$

$$\frac{d^3 u}{dx^3} = \frac{B_{11} d^4 w}{A_{11} dx^4}, \quad (14)$$

and the B.C.s at  $x = 0, a$  are:

$$w = 0, \quad (15)$$

$$N_{xx} = A_{11} \frac{du}{dx} - B_{11} \frac{d^2 w}{dx^2} = 0, \quad (16)$$

$$M_{xx} = B_{11} \frac{du}{dx} - D_{11} \frac{d^2 w}{dx^2} = 0. \quad (17)$$

Equation (13) can be written as:

$$(A_{11}D_{11} - B_{11}^2) \frac{d^4 w}{dx^4} = qA_{11}. \quad (18)$$

By applying similitude theory, the resulting similarity conditions, eqn (18), are:

$$\lambda_{A_{11}} \lambda_{D_{11}} \lambda_w = \lambda_{B_{11}}^2 \lambda_w = \lambda_{A_{11}} \lambda_x^4 \lambda_q, \quad (19)$$

or

$$\lambda_{A_{11}} \lambda_{D_{11}} = \lambda_{B_{11}}^2, \quad (20)$$

$$\lambda_w \lambda_{D_{11}} = \lambda_x^4 \lambda_q. \quad (21)$$

Similarly from eqns (14), (16) and (17) we have:

$$\lambda_{A_{11}} \lambda_u \lambda_x = \lambda_w \lambda_{B_{11}}, \quad (22)$$

$$\lambda_{B_{11}} \lambda_u \lambda_x = \lambda_w \lambda_{D_{11}}. \quad (23)$$

The condition depicted by eqn (23) does not give any new information, since it can be obtained by combining eqns (20) and (22). So, eqns (20)–(22) denote the necessary behavioral conditions for complete similarity between the scale model and its prototype.

a.1.1. Parenthesis: for better understanding the restrictions of eqn (20), consider the definition of  $A_{mn}$ ,  $B_{mn}$  and  $D_{mn}$ .

$$A_{mn} = \sum_{j=1}^N (\bar{Q}_{mn})_j (z_j - z_{j-1}),$$

$$B_{mn} = \frac{1}{2} \sum_{j=1}^N (\bar{Q}_{mn})_j (z_j^2 - z_{j-1}^2),$$

$$D_{mn} = \frac{1}{3} \sum_{j=1}^N (\bar{Q}_{mn})_j (z_j^3 - z_{j-1}^3),$$

where  $z_j$  is the coordinate of the upper surface of the  $j$ th lamina (measured from the plate reference surface). Let  $z_j = c_j h$  where  $-0.5 \leq c_j \leq 0.5$  and  $h$  is the total thickness ( $j = 0, 1, \dots, N$ ).  $(\bar{Q}_{mn})_j$ , the transformed stiffnesses for the  $j$ th lamina are given in terms of the engineering orthotropic constants and the fiber orientation angle  $\theta$ .

$$\bar{Q} = f(\theta, E_{11}, E_{22}, \nu_{12}, G_{12}).$$

This allows us to express  $A_{mn}$ ,  $B_{mn}$  and  $D_{mn}$  in terms of  $h$  and functions of all  $\bar{Q}$ ,  $N$  and the stacking sequence.

$$A_{mn} = h f_a(\bar{Q}_{mn}, N),$$

$$B_{mn} = h^2 f_b(\bar{Q}_{mn}, N),$$

$$D_{mn} = h^3 f_d(\bar{Q}_{mn}, N),$$

or as scale factors:

$$\lambda_{A_{mn}} = \lambda_h F_a(\bar{Q}_{mn}, N), \quad (24)$$

$$\lambda_{B_{mn}} = \lambda_h^2 F_b(\bar{Q}_{mn}, N), \quad (25)$$

$$\lambda_{D_{mn}} = \lambda_h^3 F_d(\bar{Q}_{mn}, N), \quad (26)$$

where:

$$F_i = \frac{f_i(\bar{Q}, N)_p}{f_i(\bar{Q}, N)_m}, \quad i = a, b, d.$$

Substitution of eqns (24)–(26) into eqn (20) yields:

$$F_a(\bar{Q}_{11}, N) F_a(\bar{Q}_{11}, N) = F_b^2(\bar{Q}_{11}, N). \quad (27)$$

Equation (27) states that the first similarity condition, eqn (20), is independent of total thickness of the plate, and it is only a function of material properties, number of plies and stacking sequence of the model and its prototype. This condition, eqn (27), is satisfied if the model and prototype are made of the same material with identical  $N$  and the same stacking sequence of the lamina.

Now, the accuracy of the derived behavioral similarity conditions, eqns (20)–(22) is evaluated analytically, in order to determine the level of confidence that can be expected in interpreting the data from the distorted model experiments (partial similarity).

Consider a cross-ply laminated E-Glass/Epoxy plate composed of 96 orthotropic layers (0/90/0/...)96 as the prototype. We desire to find the maximum deflection of the prototype by extrapolating the pertinent values of a small scale model. The model has the same stacking sequence as the prototype but with a smaller number of layers (distorted model). The prototype and its scale model have the following characteristics:

$$\text{prototype (0/90/0/...)96: } a = 90 \text{ in. } b = 100 \text{ in. } h = 0.858 \text{ in. } N = 96,$$

$$\text{model (0/90/0/...)16: } a = 5.0 \text{ in. } b = 6.139 \text{ in. } h = 0.143 \text{ in. } N = 16,$$

$$\text{scale factors: } \lambda_a = 18 \quad \lambda_b = 16.29 \quad \lambda_h = 6 \quad \lambda_N = 6.$$

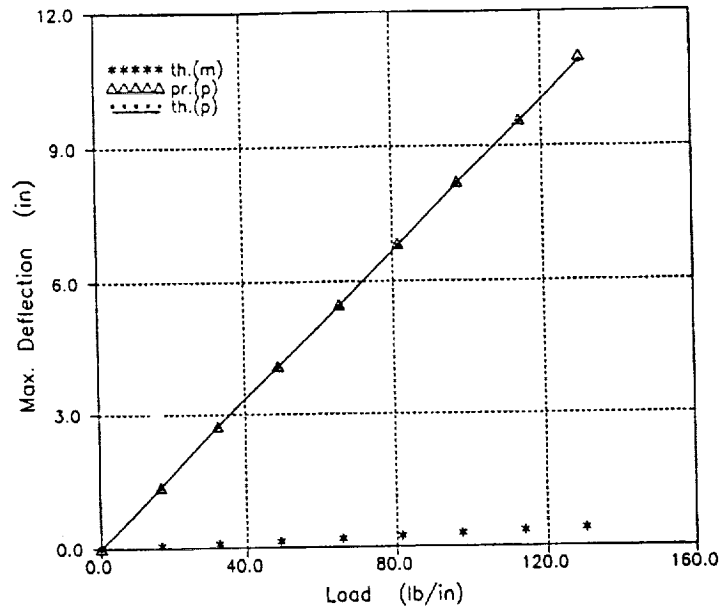


Fig. 1. Theoretical and predicted maximum deflections of prototype (0/90/0/...)<sub>96</sub> when model (0/90/0/...)<sub>16</sub> is used ( $\lambda_{E11} = \lambda_{E22} = \lambda_{\nu12} = 1$ ,  $\lambda_q = 18$ ,  $\lambda_b = \lambda_q = 16.92$ ,  $\lambda_h = \lambda_N = 6$ ).

For simplification we assume that model and prototype have the same material properties ( $\lambda_{E11} = \lambda_{E22} = \lambda_{\nu12} = 1$ ) and  $\lambda_q = \lambda_b$ . By employing only the similarity condition of eqn (21) (note that  $\lambda_p = \lambda_x \lambda_q$ ; therefore the condition becomes  $\lambda_w \lambda_{D11} = \lambda_x^3 \lambda_p$ ) the theoretical maximum deflections of the model are projected in order to predict the maximum deflections of the prototype. Figure 1 presents the theoretical and predicted maximum deflections of the prototype and corresponding theoretical values of the scale model. The derived scaling laws can be used with a high level of accuracy in predicting the prototype behavior. Note that the model was designed by employing the free scaling factors (partial similarity).

a.2. *Stress analysis.* For the  $k$ th lamina, the normal stress in terms of the strains and curvatures (cylindrical bending) is:

$$\sigma_{xx}^{(k)} = \bar{Q}_{xx}^{(k)}(\epsilon_{xx}^0 + z k_{xx}), \quad (28)$$

where  $\epsilon_{xx}^0$  is the extensional strain on the reference surface ( $z = 0$ ) and  $k_{xx}$  represents the change in curvature of the reference surface.

By substituting the expressions for  $\epsilon_{xx}^0$  and  $k_{xx}$

$$\sigma_{xx}^{(k)} = \bar{Q}_{11}^{(k)}(u_{,x} + \frac{1}{2} w_{,x}^2 - z w_{,xx}). \quad (29)$$

Applying similitude theory for the normal stress,  $\sigma_{xx}$ ,

$$\lambda_{\sigma_{xx}^{(k)}} = \lambda_{\bar{Q}_{11}^{(k)}} \left( \frac{\lambda_u}{\lambda_x} + \frac{\lambda_w^2}{\lambda_x^2} - \lambda_z \frac{\lambda_w}{\lambda_x^2} \right). \quad (30)$$

The resulting similarity conditions are:

$$\lambda_{\sigma_{xx}^{(k)}} = \lambda_{\bar{Q}_{11}^{(k)}} \lambda_u \lambda_x^{-1}, \quad (31)$$

$$\lambda_{\sigma_{xx}^{(k)}} = \lambda_{\bar{Q}_{11}^{(k)}} \lambda_w^2 \lambda_x^{-2}, \quad (32)$$

$$\lambda_{\sigma_{xx}^{(k)}} = \lambda_{\bar{Q}_{11}^{(k)}} \lambda_z \lambda_w \lambda_x^{-2}, \quad (33)$$

where  $\lambda_w = \lambda_x^3 \lambda_p \lambda_{D11}^{-1}$  and  $\lambda_u = \lambda_w \lambda_x^{-1} \lambda_{B11} \lambda_{A11}^{-1}$  [see eqns (21) and (22)].

For complete similarity, eqns (31)–(33) give the same result. However, for the distorted model each similarity condition gives different results. To find which one of eqns (31)–(33) gives the best prediction for the prototype behavior (partial similarity), the

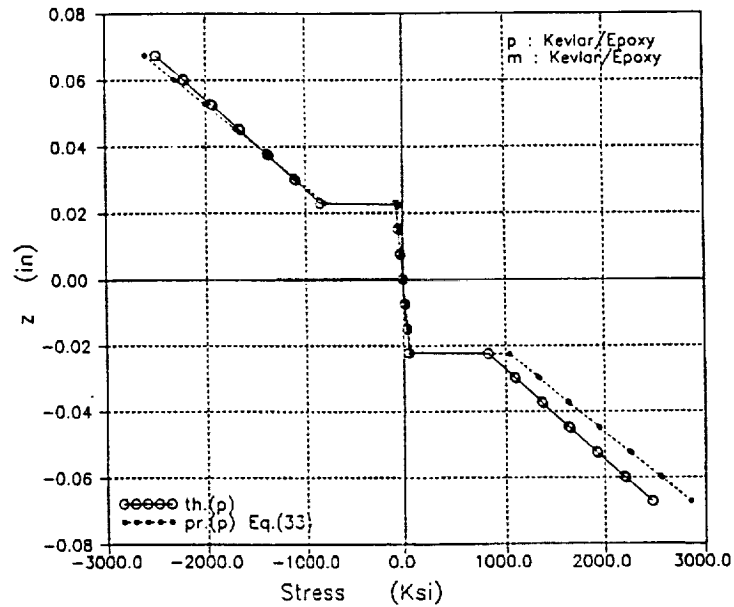


Fig. 2. Predicted and theoretical normal stress  $\sigma_{xx}$  distributions in various layers of the prototype  $(0_6/90_6/0_6)$  when  $(0_6/90_6/0_6)_{18}$  is used as model.

theoretical stresses of the model are projected with each condition and compared to theoretical stresses of the prototype.

Figures 2-4 present the predicted and theoretical distributions of the normal stress  $\sigma_{xx}$  in various layers of the prototype for a cylindrical bending test. It is observed that the predicted stresses by eqn (33) agree very well with the theoretical results. Equation (32) cannot predict the behavior of the prototype accurately. Equation (31) is not a suitable similarity condition, since its predicted data does not match the theoretical results. The figures do not include the predicted stresses using eqns (31) and (32). This is purposely done in order to simplify the figures. Since the results came from a three-point bending case, it is expected that the similarity condition based on bending normal stresses, eqn (33), would yield accurate stress predictions.

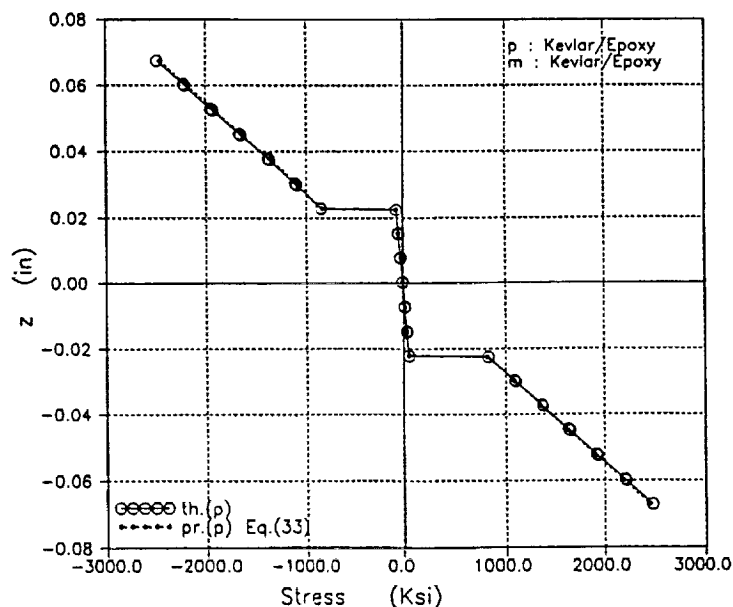


Fig. 3. Predicted and theoretical normal stress  $\sigma_{xx}$  distributions in various layers of the prototype  $(0_6/90_6/0_6)$  when  $(0_4/90_4/0_4/90_4/0_4)$  is used as model.



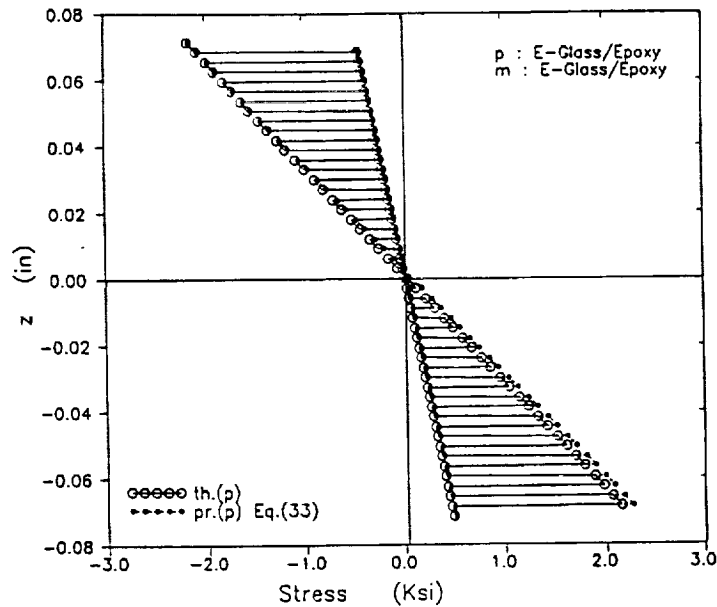


Fig. 4. Predicted and theoretical normal stress  $\sigma_{xx}$  distributions in various layers of the prototype  $(0/90/0/\dots)_{48}$  when  $(0/90/0/\dots)_{16}$  is used as model.

It is necessary to note that the stress distribution in various layers of the prototype is completely different from the stress distribution in the model. But, the derived similarity condition, eqn (33), can be used successfully to predict the stress distribution in the prototype.

#### b. Buckling of symmetric laminated cross-ply rectangular plates

Consider the case of cross-ply symmetric laminated plates ( $B_{ij} = 0$ ,  $D_{16} = D_{26} = A_{16} = A_{26} = 0$ ). The plates are subjected to inplane uniaxial compression in the  $x$  direction ( $\bar{N}_x$ ).

The buckling loads are described only by one differential equation:

$$D_{11} w_{,xxxx}^0 + 2\bar{D}_{12} w_{,xxyy}^0 + D_{22} w_{,yyyy}^0 - \bar{N}_x w_{,xx}^0 = 0, \quad (34)$$

where  $\bar{D}_{12} = D_{12} + 2D_{66}$ .

For simply supported plates, the boundary conditions are:

at  $x = 0, a$

$$w = 0, \quad w_{,xx}^0 = 0,$$

at  $y = 0, b$

$$w = 0, \quad w_{,yy}^0 = 0.$$

The solution:

$$w = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right),$$

satisfies the boundary conditions and governing differential equation if:

$$\bar{N}_x = \pi^2 \left[ D_{11} \left(\frac{m}{a}\right)^2 + 2\bar{D}_{12} \left(\frac{n}{b}\right)^2 + D_{22} \left(\frac{n}{b}\right)^4 \left(\frac{a}{m}\right)^2 \right]. \quad (35)$$

By applying similitude theory to eqn (35):

$$\lambda_{N_x} = \lambda_{D_{11}} \frac{\lambda_m^2}{\lambda_a^2} = \lambda_{\bar{D}_{12}} \frac{\lambda_n^2}{\lambda_b^2} = \lambda_{D_{22}} \frac{\lambda_n^4 \lambda_a^2}{\lambda_b^4 \lambda_m^2}, \quad (36)$$

which yields to the following scaling laws:

$$\lambda_{N_x} = \lambda_{D_{11}} \frac{\lambda_m^2}{\lambda_a^2}, \quad (37)$$

$$\lambda_{R_1} = \frac{\lambda_m}{\lambda_n} \left( \frac{\lambda_{D_{11}}}{\lambda_{D_{12}}} \right)^{0.25}, \quad (38)$$

$$\lambda_{R_2} = \frac{\lambda_m}{\lambda_n} \left( \frac{\lambda_{D_{11}}}{\lambda_{D_{22}}} \right)^{0.25}. \quad (39)$$

Boundary conditions do not yield any similarity condition. Equation (37) can be written as:

$$\lambda_{K_1} = \frac{\lambda_{D_{11}} \lambda_m^2}{\lambda_{D_{22}} \lambda_R^2}, \quad (40)$$

where:

$$K_1 = \frac{N_x b^2}{\pi^2 D_{22}} \Rightarrow \lambda_{K_1} = \frac{\lambda_{D_{11}} \lambda_b^2 \lambda_m^2}{\lambda_{D_{22}} \lambda_a^2}.$$

Conditions (38) and (39) are considered as *design scaling laws* and condition (40) is considered as *response scaling law*.

b.1. *Complete similarity.* The necessary condition for complete similarity between the model and its prototype is that all the design scaling laws predict the behavior of the prototype with the same accuracy. In other words  $\lambda_{R_1} = \lambda_{R_2}$ . This equality is satisfied if [see eqns (38) and (39)]:

$$\lambda_{D_{12}} = \lambda_{D_{22}}, \quad \lambda_{D_{11}} \lambda_{D_{22}} = \lambda_{D_{12}}^2. \quad (41)$$

From Tsai (1964), the bending stiffnesses  $D_{mn}$  can be expressed as a function of the total number of layers,  $N$ , the cross-ply ratio,  $M$ , and stiffness ratio,  $F$ ,

$$D_{11} = [(F - 1)\psi + 1] \frac{h^3}{12} Q_{11},$$

$$D_{22} = [(1 - F)\psi + F] \frac{h^3}{12} Q_{11},$$

$$D_{12} = \frac{h^3}{12} Q_{12},$$

where  $F = E_{22}/E_{11} = Q_{22}/Q_{11}$  and

$$\psi = \frac{1}{(1 + M)^3} + \frac{M(N - 3)[M(N - 1) + 2(N + 1)]}{(N^2 - 1)(1 + M)^3},$$

where  $M$  is the cross-ply ratio and  $N$  is the total number of plies. For the common special case of symmetric cross-ply laminates  $(0/90/0/\dots)_n$ , in which the laminae are all the same thickness and have the same material properties:

$$M = \frac{N + 1}{N - 1}.$$

Substituting into eqn (41)

$$\left[ \frac{(F_p - 1)\psi_p + 1}{(F_m - 1)\psi_m + 1} \right] \left[ \frac{(1 - F_p)\psi_p + F_p}{(1 - F_m)\psi_m + F_m} \right] \lambda_{Q_{11}}^2 = \lambda_{Q_{12}}^2. \quad (42)$$

In general, by choosing the model material and using eqn (42), the number of plies of model ( $N_m$ ) can be determined. Since  $N_m$  must be an integer, it is difficult to satisfy eqn (42), therefore partial similarity with a distorted model is pursued.

For simplification we assume that model and prototype have the same material properties ( $\lambda_{E_{11}} = \lambda_{E_{22}} = \lambda_{\nu_{12}} = 1$ ). If the model and prototype have the same material properties then  $\lambda_{Q_{11}} = \lambda_{Q_{12}} = 1$  and  $F_p = F_m = F$  and eqn (42) can be simplified as:

$$[(F - 1)\psi_p + 1][(1 - F)\psi_p + F] = [(F - 1)\psi_m + 1][(1 - F)\psi_m + F],$$

$$\psi_p \cdot (\psi_p - 1) = \psi_m \cdot (\psi_m - 1),$$

$$f(N_p, N_m) = \frac{\psi_p \cdot (\psi_p - 1)}{\psi_m \cdot (\psi_m - 1)} = 1.$$

The numerical values of  $f(N_p, N_m)$  are plotted over a large range of  $N_m$  for several  $N_p$  in Fig. 5. From Fig. 5, it is verified that complete similarity is achieved if  $\lambda_N = 1$ . It is also important to notice that, as the number of plies of prototype increases,  $\lambda_N \approx 1$  also satisfies the condition  $f(N_p, N_m) = 1$  and complete similarity is achieved. Since the number of plies,  $N_p$  and  $N_m$  are integer numbers this condition exists for large  $N_p$ .

So far we proved in the special case when  $\lambda_N \approx 1$ , condition  $f(N_p, N_m) = 1$  is satisfied and complete similarity is achieved. However, there are some constraints, such as the geometry of the model, the model material, the number of plies and the stacking sequence of laminates. Since this still appears to be restrictive, we proceed with the determination of distorted models, for which some of these restrictions can be relaxed.

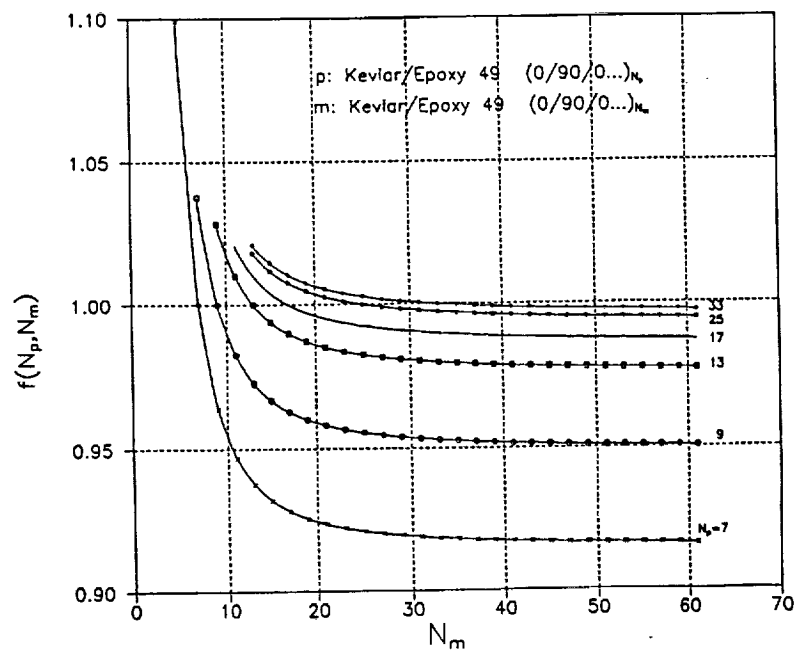


Fig. 5. Sensitivity of complete similarity condition  $\lambda_{D_{11}} \lambda_{D_{12}} = \lambda_{D_{12}}^2$  for different  $\lambda_N$  ( $f(N_p, N_m) = \lambda_{D_{11}} \lambda_{D_{12}} / \lambda_{D_{12}}^2 = 1$  complete similarity).

**b.2. Partial similarity.** When at least one of the design scaling laws cannot be satisfied, partial similarity is achieved. In this case, since each parameter has different influence on the response of the system, the resulting design scaling laws have different influence on the accuracy of the predicted response. By understanding the effect of the various parameters and accuracy of the design scaling laws over desired intervals, the design scaling laws which have the least accurate prediction can be chosen as the "right" type of distortion.

The choice of the right type of distortion is investigated as follows. In each case, all of the model parameters except one, are chosen to be identical to its prototype. Then, the effect of this relaxation for a wide range of this parameter is investigated.

b.2.1. Number of plies: since we assume all the laminae to have equal thickness, the distortion in thickness is the same as the distortion in number of plies ( $h_{\text{total}} = N \cdot h_{\text{laminae}}$ ). In other words, thickness is only a function of the number of plies. Consider that the model and prototype have the same material properties (Kevlar/Epoxy 49) with a different number of layers ( $N_m \neq N_p$ ). Figure 6 presents the per cent of discrepancy between theoretical and predicted values of the normalized critical load ( $K_1$ ) for simply supported rectangular plates. In these cases, the prototype is a laminated cross-ply  $(0/90/0/\dots)_{101}$  square plate with  $R_p = 1$ . The accuracy of the designed distorted models with the same stacking sequence but with a different number of layers  $[(0/90/0/\dots)_i]$  is investigated. The  $R_m$  is determined by using both design scaling laws [eqns (38) and (39)]. Figure 6 shows that, as the number of plies of the model increases, the accuracy of the model increases very quickly. Models with  $N = 1$ , or 3 do not have acceptable accuracy. Equation (39) is the best design scaling law, especially for a model with  $5 \leq N_m < 40$ . For  $N_m > 40$  all conditions yield the same accuracy.

The accuracy of a model with  $N = 13$  in predicting the buckling behavior of prototypes with  $N \geq 101$  is also investigated. It is shown that, the model with  $N_m = 13$  can predict the critical load of any prototype with  $N_p = 101, \dots, 500$  with the same accuracy as prototype with  $N_p = 101$ . In other words, the accuracy is independent of  $N_p$  when  $N_p \geq 101$ .

This study indicates that a distorted model with a smaller number of layers can predict the critical load of its prototype with good accuracy.

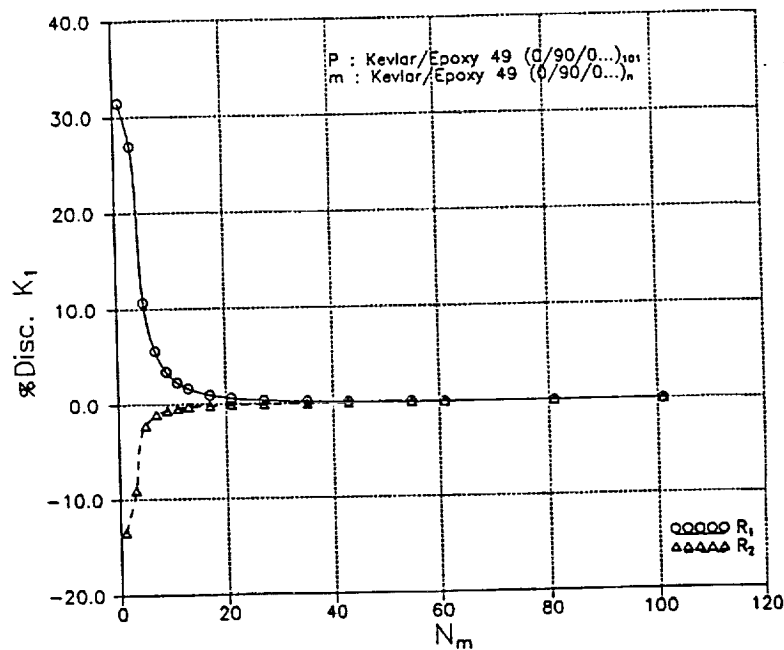


Fig. 6. % discrepancy of normalized buckling loads ( $K_1$ ) when  $N_m \leq N_p$  (distortion in  $N$ ).

b.2.2. Material: now we consider the distortion in model material. For this purpose two different groups are considered: isotropic materials (which include metals and plastics) and fiber-reinforced composites. In all cases the prototype is considered to be an orthotropic laminated plate.

The model and its prototype have the same stacking sequence, number of plies  $[(0/90/0/\dots)_{13}]$  and aspect ratio. Figure 7 presents the per cent of discrepancy when the model and prototype have different material properties. For the Kevlar/Epoxy prototype, a Boron/Epoxy, a Boron/Polymide and most of the Graphite/Epoxy's can be used as the model material and vice versa. However S-Glass/Epoxy is not a good choice for predicting a Kevlar/Epoxy prototype and vice versa. The design scaling laws of eqn (38) yield the best accuracy.

## Model Materials

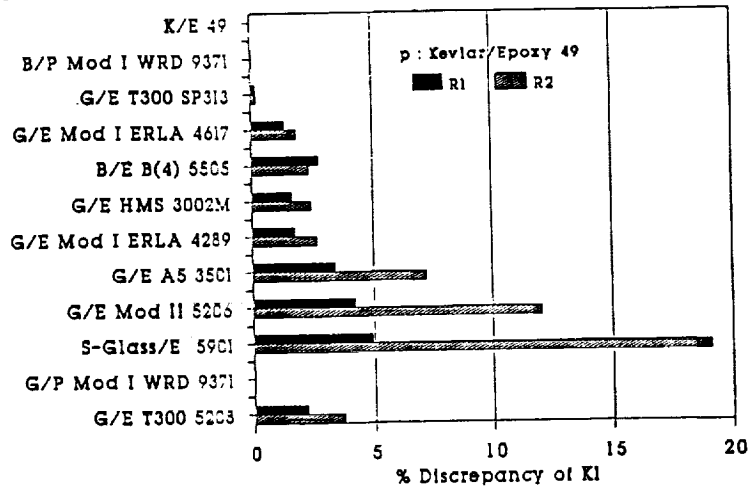


Fig. 7. % discrepancy of normalized buckling loads ( $K_1$ ) for different composite materials. Model and prototype have the same stacking sequences  $(0/90/0/\dots)_{13}$ .

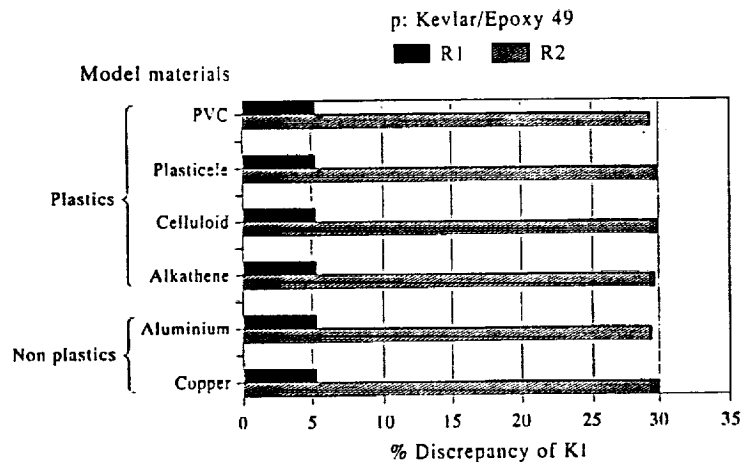


Fig. 8. % discrepancy of normalized buckling load for different isotropic materials. Model and prototype have the same stacking sequences  $(0/90/0/\dots)_{13}$ .

Since plastics are used extensively for the experimental study of the behavior of the structures, the possibility of a plastic model or in general a model with isotropic materials is considered. Figure 8 presents the per cent of discrepancies for models with isotropic materials. The prototype is Kevlar/Epoxy plate  $[(0/90/0/\dots)_{13}]$ . Almost all of the plastics, copper and aluminum which are used, give the same accuracy for the design scaling law of eqn (39) (30% discrepancy). But the design scaling law of eqn (38) yields better accuracy (less than 5.3%).

## DISCUSSION

An extensive study based on analytical investigations has been conducted in order to establish the applicability of similitude theory to simple structural elements. Theory of similitude is used to design scale models for orthotropic laminated beamplates and to predict the behavior of the prototype, with reasonable accuracy. Similarly data of scale models are projected to predict prototype behavior. Even for models with different numbers of plies and stacking sequence of layers (distorted models) the predicted data are well-matched with full scale prototype data.

By establishing similarity conditions the model parameters are specified. First, there is a need to verify the derived similarity conditions. The verification should be based on

exact analytical solution of the system. Furthermore, a validation procedure based on experimental data of two or three small scale models of the actual system (prototype) should be conducted. For example, at least two different small scale models of the prototype should be designed and tested. One of these models should serve as prototype and the other one as scale model. The experimental data of the model should be used to predict the behavior of the other one. This increases the confidence factor.

#### CONCLUSIONS AND RECOMMENDATIONS

While several experimental studies can be done on prototype or large scale models, size constraints will limit experimentation on complete structures in most cases to small scale models. Small scale models can be used as a complement for analytical and computational investigations in solving the design problems of complex structures. The investigation in this study indicates that the use of small scale models can predict the behavior of the prototype very well.

Partial similarity based on direct use of governing equations is more convenient than dimensional analysis, because additional relationships are not needed and the derived similarity conditions are based on satisfaction of the field equations of the system.

In the present study, similitude theory was used by employing systems for which the experimental results are not known. In this case, one system was considered to be the prototype and another its scale model. Then through the use of the proper scaling laws the theoretical data of the model were used to predict the behavior of the prototype. Success was measured by comparing the predicted behavior to the analytical results.

Some recommendations for future research:

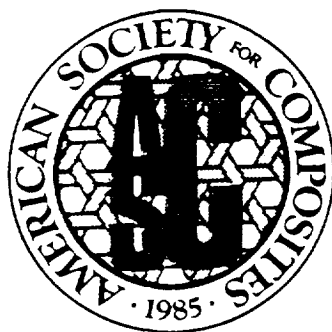
- develop the method for designing and employing scale models for more complex systems, i.e. stiffened and/or laminated curved configurations;
- one of the major problems associated with inelastic analysis of the small scale model is the effect of size. A need exists to evaluate the size effect in material behavior especially for geometries with higher scale factors. This is also true in dealing with establishing the strength of laminated structures, since strength is affected by the accumulation of damage.

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# Structural Similitude and Scaling Laws for Cross-Ply Laminated Plates

G. J. SIMITSES† AND J. REZAEPAZHAND\*

**Abstract.** The increasing use of laminated composite components for a wide variety of applications in aerospace, mechanical and other branches of engineering requires extensive experimental evaluation of any new design. Thus, it is extremely useful if a full-scale structure can be replaced by a similar scaled-down model which is much easier to work with. The objective of this study is to investigate problems associated with design of scaled models. Similitude theory is employed to develop the necessary similarity conditions. Both complete and partial similarity are discussed. The procedure consists of systematically observing the effect of each parameter and corresponding scaling laws. Then acceptable intervals and limitations for these parameters and scaling laws are discussed. In each case, a set of valid scaling factors and corresponding response scaling laws that accurately predict the response of prototypes from experimental models is introduced. Particular emphasis is placed on the cases of buckling of rectangular cross-ply laminated plates under uniaxial compressive and shear loads. This analytical study indicates that distorted models with a different number of layers, material properties, and geometries than those of the prototype can predict the behavior of the prototype with good accuracy.

## Nomenclature

$a$	plate length	$Q_{ij}, \bar{Q}_{ij}$	lamina stiffness elements
$A_{ij}$	laminate extensional stiffnesses	$R$	aspect ratio
$b$	plate width	$u, v, w$	reference surface displacements
$B_{ij}$	laminate coupling stiffnesses	$t$	ply thickness
$D_{ij}$	laminate flexural stiffnesses	$\lambda_i$	scale factors
$E_{ij}$	Young's moduli of elasticity	$\nu_{ij}$	Poisson's ratios
$h$	total laminate thickness	$m$	model
$K_{xx}, K_s, \bar{K}_s$	non-dimensional critical loads	$p$	prototype
$M_x, M_y$	moment resultants	$pr.$	predicted
$\bar{N}_{xx}, \bar{N}_{yy}$	inplane normal loads	$th.$	theoretical
$\bar{N}_{xy}$	inplane shear load		

## INTRODUCTION

The last step in the design process, before going to production, is the verification of

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the design. This step necessitates the production of large components and full scale prototypes in order to test component and system analytical predictions and verify strength and performance requirements under the worst loading conditions that the system is expected to encounter in service. A scaled-down (by a large factor) model, *scale model*, which closely represents the structural behavior of the full-scale system, *prototype*, can prove to be an extremely beneficial tool. This possible development must be based on the existence of certain structural parameters that control the behavior of the structural system when acted upon by static and/or dynamic loads. If such structural parameters exist, a scaled-down replica can be built, which will duplicate the response of the full-scale system. The two systems are then said to be structurally similar. The term, then, that best describes this similarity is *Structural Similitude*.

Due to special characteristics of advanced reinforced composite materials, they have been used extensively in weight efficient aerospace structures. Since reinforced composite components require efficiency and wisdom in design, sophistication and accuracy in analysis, and numerous and careful experimental evaluations, there is a growing interest in small scale model testing[4].

By applying similitude theory, we try to find a set of conditions between two similar structural systems (scaling laws). Later, these conditions can be used to design a model, the experimental data of which can be projected in order to predict the behavior of the prototype.

The objectives of the investigation described herein are:

- create necessary similarity conditions in order to design an accurate distorted model
  - distortion in stacking sequence and number of plies ( $N$ )
    - \* ply - level scaling
    - \* sublaminates - level scaling
  - distortion in material properties  $E_{ij}, \nu_{ij}, \rho$
- evaluate the derived similarity conditions analytically.

Similarity conditions provide the relationship between model and its prototype, and can be used to extrapolate the experimental data of a small and less expensive model in order to predict the behavior of the prototype. This study presents the applicability of small scale models, especially distorted models, in analyzing the elastic behavior of cross-ply laminated plates. Furthermore, it is assumed that the laminates are free of damage (delaminations, matrix cracking, fiber breaks, etc.).

In this study, we consider only the procedure that is based on the direct use of the "governing equations". This method is more convenient than dimensional analysis, since the resulting similarity conditions are more specific and the relationships among variables are forced by the governing equations of the system.

Often complete similarity is difficult to achieve or even undesirable. When at least one of the similarity conditions can not be satisfied, partial similarity is achieved. In this case, the model which has some relaxation in similarity conditions is called a *distorted model*. Distorted models are more practical, since relaxation of each similarity condition

eliminates some restrictions on the model design. These relaxations in the relationship between two systems cause model behavior to be different from that of the prototype. Since each variable has different influence on the response of the system, the resulting similarity conditions have different influence. By understanding the effect of variables and similarity conditions over desired intervals, the similarity conditions which have the least influence can be neglected without introducing significant error[3].

### Buckling of Symmetric Laminated Cross-Ply Rectangular Plates

Consider symmetric cross-ply laminated plates ( $B_{ij} = 0$ ,  $D_{16} = D_{26} = A_{16} = A_{26} = 0$ ). The plates are subjected to inplane normal and shear loads ( $\bar{N}_{xx}$ ,  $\bar{N}_{yy}$ ,  $\bar{N}_{xy}$ ). The governing differential equations for buckling and vibration of symmetric cross-ply rectangular plates are as follow [1]

$$A_{11}u_{,xx}^0 + A_{12}v_{,yx}^0 + A_{66}(u_{,yy}^0 + v_{,xy}^0) = 0 \quad (1)$$

$$A_{66}(u_{,yx}^0 + v_{,xx}^0) + A_{12}u_{,xy}^0 + A_{22}v_{,yy}^0 = 0 \quad (2)$$

$$D_{11}w_{,xxxx}^0 + 2\bar{D}_{12}w_{,xxyy}^0 + D_{22}w_{,yyyy}^0 - \bar{N}_{xx}w_{,xx}^0 - \bar{N}_{yy}w_{,yy}^0 - \bar{N}_{xy}w_{,xy}^0 = \rho w_{,tt}^0 \quad (3)$$

where  $\bar{D}_{12} = D_{12} + 2D_{66}$

For simply supported plate, the approximate boundary conditions are

$$\text{at } x = 0, a \quad w = 0, \quad M_x = -D_{11}w_{,xx}^0 = 0$$

$$\text{at } y = 0, b \quad w = 0, \quad M_y = -D_{22}w_{,yy}^0 = 0$$

**I. Uniaxial Load :** Consider the plates to be subjected to inplane uniaxial compression load in  $x$  direction ( $\bar{N}_{xx}$ ). The buckling differential equation is:

$$D_{11}w_{,xxxx}^0 + 2\bar{D}_{12}w_{,xxyy}^0 + D_{22}w_{,yyyy}^0 - \bar{N}_{xx}w_{,xx}^0 = 0 \quad (4)$$

The solution

$$w = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

satisfies the boundary conditions. Then,

$$\bar{N}_{xx} = \pi^2 \left[ D_{11} \left(\frac{m}{a}\right)^2 + 2\bar{D}_{12} \left(\frac{n}{b}\right)^2 + D_{22} \left(\frac{n}{b}\right)^4 \left(\frac{a}{m}\right)^2 \right] \quad (5)$$

By applying similitude theory to Eq. (5)

$$\lambda_{\bar{N}_{xx}} = \lambda_{D_{11}} \frac{\lambda_m^2}{\lambda_a^2} = \lambda_{D_{12}} \frac{\lambda_n^2}{\lambda_b^2} = \lambda_{D_{22}} \frac{\lambda_n^4 \lambda_a^2}{\lambda_b^4 \lambda_m^2} \quad (6)$$

which yields to following scaling laws

$$\lambda_{K_{xx}} = \frac{\lambda_{D_{11}}}{\lambda_{E_{22}} \lambda_h^3} \frac{\lambda_m^2}{\lambda_R^2} \quad (7)$$

$$\lambda_{K_{xx}} = \lambda_n^2 \frac{\lambda_{D_{12}}}{\lambda_{E_{22}} \lambda_h^3} \quad (8)$$

$$\lambda_{K_{xx}} = \frac{\lambda_{D_{22}}}{\lambda_{E_{22}} \lambda_h^3} \lambda_n^4 \lambda_R^2 \quad (9)$$

where  $K_{xx} = \frac{\bar{N}_{xx} b^2}{E_{22} h^3}$

Boundary conditions do not give any similarity condition. These conditions, Eqs. (7)–(9), involve response ( $\lambda_{K_{xx}}$ ,  $\lambda_m$ ,  $\lambda_n$ ) and structural geometric parameters ( $\lambda_{D_{ij}}$ ,  $\lambda_h$ ,  $\lambda_R$ ).

**II. Shear Buckling:** We now consider a simply supported plate which is subjected to the an in-plane shear stress ( $\bar{N}_{xx} = \bar{N}_{yy} = w_{,tt}^0 = 0$ ).

$$D_{11} w_{,xxxx}^0 + 2\bar{D}_{12} w_{,xxyy}^0 + D_{22} w_{,yyyy}^0 - 2\bar{N}_{xy} w_{,xy}^0 = 0 \quad (10)$$

The solution of the form

$$w^0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

satisfies all B.C.'s, but does not satisfy the buckling equation, Eq. (10). Use of the Galerkin procedure, yields

$$\left( \frac{D_{11}}{E_{22} h^3} \frac{m^4}{R^3} + 2 \frac{\bar{D}_{12}}{E_{22} h^3} \frac{m^2 n^2}{R} + \frac{D_{22}}{E_{22} h^3} R n^4 \right) A_{mn} = K_s \frac{32mn}{\pi^4} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{pq} Q_{mnpq} \quad (11)$$

for  $m, n = 1, 2, \dots, \infty$ ; subject to the constraints  $m \pm p = \text{odd}$  and  $n \pm q = \text{odd}$ .

$$K_s = \frac{\bar{N}_{xy} b^2}{E_{22} h^3}, \quad R = \frac{a}{b}, \quad Q_{mnpq} = \frac{pq}{(m^2 - p^2)(n^2 - q^2)}$$

Applying similitude theory to Eq. (11)

$$\frac{\lambda_{D_{11}}}{\lambda_{E_{22}} \lambda_h^3} \frac{\lambda_m^4}{\lambda_R^3} = \frac{\lambda_{D_{12}}}{\lambda_{E_{22}} \lambda_h^3} \frac{\lambda_m^2 \lambda_n^2}{\lambda_R} = \frac{\lambda_{D_{22}}}{\lambda_{E_{22}} \lambda_h^3} \lambda_R \lambda_n^4 = \frac{\lambda_{K_s}}{\lambda_{A_{mn}}} \lambda_m \lambda_n \lambda_\Omega \quad (12)$$

where

$$\Omega = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{pq} Q_{mnpq}$$

Parenthesis:

$m, n, p$ , and  $q$  are integers which depend on the number of terms needed to approximate well the buckling mode shape (Symmetric/Anti-symmetric). By assuming the same aspect ratio for the model and its prototype ( $\lambda_R = 1$ ) and similar construction, model and prototype both can be well approximated by the same number of terms in the series with the same contribution of terms to the buckling mode and thus

$$\lambda_m = \lambda_n = \lambda_p = \lambda_q = 1 \implies \lambda_\Omega = 1, \lambda_{A_{mn}} = 1$$

Eq.( 12) yields to the following scaling laws

$$\lambda_{K_s} = \frac{\lambda_{D_{11}}}{\lambda_{E_{22}} \lambda_h^3 \lambda_R^3} \quad (13)$$

$$\lambda_{K_s} = \frac{\lambda_{D_{12}}}{\lambda_{E_{22}} \lambda_h^3 \lambda_R^3} \quad (14)$$

$$\lambda_{K_s} = \frac{\lambda_{D_{22}}}{\lambda_{E_{22}} \lambda_h^3} \lambda_R \quad (15)$$

These conditions, Eqs.( 13)–( 15), also involve response and structural geometric parameters.

### Complete Similarity :

The necessary condition for complete similarity between the model and its prototype is that all scaling laws be satisfied. This requirement yields

$$\lambda_{D_{11}} = \lambda_{D_{12}} = \lambda_{D_{22}} \quad (16)$$

By inspection it can be seen that these conditions, Eqs.( 16), are independent of ply thickness and they only depend on material properties and number of plies. So, two plates with different ply thickness but the same stacking sequences (i.e.(0/90)<sub>s</sub> and (0<sub>n</sub>/90<sub>n</sub>)<sub>s</sub>) satisfy Eqs.( 16). This is called ply-level scaling and it is the easiest way to achieved complete similarity. Table 1 presents the ply-level scaling for cross-ply plates under inplane shear loads.

So far, we have shown that in the special case of ply-level scaling similarity can be achieved. However, there are some constraints, in designing the model. These constraints involve the geometry of the model, the model material, the number of plies and the stacking sequence of laminates. Since this still appears to be restrictive, we allow the use of distortion in the design of the model.

Table 1 Comparison of shear buckling loads of Kevlar/Epoxxy plates with ply - level scaling(complete similarity).

model	$K_s = \frac{N_{xy} b^2}{E_{22} h^3}$			%Disc.	
Configuration	model	prototype	predicted	th.(p)&pr.(p)	th.(p)&th.(m)
(0 <sub>2</sub> /90 <sub>2</sub> ) <sub>s</sub>	32.74	32.74	32.74	0.0	0.0
(0 <sub>10</sub> /90 <sub>10</sub> ) <sub>s</sub>	32.74	32.74	32.74	0.0	0.0
(0 <sub>20</sub> /90 <sub>20</sub> ) <sub>s</sub>	32.74	32.74	32.74	0.0	0.0

$$\% \text{Disc.}(th.\&pr.) = 100 \times \frac{|theory - predicted|}{theory}$$

Partial Similarity : When at least one of the design scaling laws cannot be satisfied, partial similarity is achieved. By understanding the effect of parameters and accuracy of the scaling laws over desired intervals, the scaling laws which yield the most

accurate prediction for the prototype can be chosen as corresponding to the "right" type of distortion.

The choice of the "right" type of distortion is investigated as follows. In each case, all of the model parameters except one, are chosen to be identical to its prototype. Then, the effect of this relaxation for a wide range of this parameter is investigated.

**Number of Plies :** There are three ways to scale down the number of plies in a model. a) ply-level scaling  $((0_n/90_n)_s)$  b) sublamine level scaling  $[2]((0/90)_{n,s})$  and c) general reduction of plies. The ply-level scaling leads to complete similarity (as already discussed). But the two other methods yield partial similarity. Figures 1 and 2 present the compressive and shear buckling loads for models with different number of plies than those of the prototype and the predicted loads by using scaling laws depicted by Eqs. (8) and (14). Both sublamine level and general scaling are presented. It is shown that all models (except for  $(0/90/0)$ ) can predict the prototype behavior with excellent accuracy.

Table 2 Accuracy of models with sublamine level scaling ( $N_p \neq N_m$ ) using non-dimensional load  $K_s$ ,  $\lambda_{E_{ij}} = \lambda_{\nu_{ij}} = 1$ ;  $(0/90)_{20s}$ .

model	$K_s = \frac{N_{xy}b^2}{E_{22}h^3}$			%Disc.	
Configuration	model	prototype	predicted	th.(p)&pr.(p)	th.(p)&th.(m)
$(0/90)_{2s}$	32.740	34.040	32.740	3.82	3.82
$(0/90)_{5s}$	34.009	34.040	34.009	0.09	0.09
$(0/90)_{10s}$	34.034	34.040	34.034	0.02	0.02

Table 3 Accuracy of models with sublamine level scaling ( $N_p \neq N_m$ ), using non-dimensional load  $\bar{K}_s$ ,  $\lambda_{E_{ij}} = \lambda_{\nu_{ij}} = 1$ ;  $(0/90)_{20s}$ .

model	$\bar{K}_s = \frac{N_{xy}b^2}{D_{22}}$			%Disc.	
Configuration	model	prototype	predicted	th.(p)&pr.(p)	th.(p)&th.(m)
$(0/90)_{2s}$	150.20	56.763	54.594	3.82	164.61
$(0/90)_{5s}$	63.045	56.763	56.709	0.10	11.07
$(0/90)_{10s}$	58.718	56.763	56.752	0.02	3.45

**Material :** Now we consider distortion in model material. For this purpose two different groups are considered: Isotropic materials (which include metals and plastics), and fiber reinforced composites. In all of these cases the prototype is an orthotropic laminated plate.

For the composite model, model and prototype have the same stacking sequence, number of plies  $((0/90/0\ldots)_{13})$  and aspect ratio. Figures 3 and 4 present theoretical and predicted buckling loads of prototype and theoretical ones of the models for some typical composite materials. For the Kevlar/Epoxy prototype a Boron/Epoxy, Boron/Polymide, and most of Graphite/Epoxy's can be used as the model material or vice versa. But Glass/Epoxy is not a good choice for predicting a Kevlar/Epoxy prototype or vice versa.

Since plastics are used extensively in experimental studies of the behavior of the structures, the possibility of a plastic model or in general a model with isotropic materials is considered. For isotropic materials, the assumption of  $\lambda_R = 1$  yields a model which cannot predict accurately the behavior of the prototype. By choosing  $R_m$  as a design parameter we are able to find isotropic models which yields excellent accuracy. Scaling laws depicted by Eq.( 8) and Eq.( 14), yield acceptable aspect ratios for the models. Tables 4 and 5 present theoretical and predicted buckling loads for prototypes when the corresponding models are isotropic material.

Table 4 *Relaxation in material properties by using isotropic model.*  
*Prototype Is Kevlar/Epoxy (0/90)<sub>20s</sub>.*

model		$K_{xx} = \frac{N_{xx}b^2}{E_{xx}h^3}$			%Disc.	
material	$R_m$	model	prototype	predicted	th.(p)&pr.(p)	th.(p)&th.(m)
Aluminum	0.705	4.174	14.16	14.16	0.0	70.51
Brass	0.705	4.143	14.16	14.16	0.0	50.80
Copper	0.705	4.149	14.16	14.16	0.0	70.69
Steel	0.705	4.045	14.16	14.16	0.0	71.36
PVC	0.705	4.374	14.16	14.16	0.0	69.1
Polyethylene	0.705	4.760	14.16	14.16	0.0	66.38

Table 5 *Relaxation in material properties by using isotropic model. Prototype is Kevlar/Epoxy (0/90)<sub>20s</sub>.*

model		$K_y = \frac{N_{xy}b^2}{E_{xx}h^3}$			%Disc.	
material	$R_m$	model	prototype	predicted	th.(p)&pr.(p)	th.(p)&th.(m)
Aluminum	0.627	20.22	34.04	34.04	0.0	40.61
Brass	0.627	20.07	34.04	34.04	0.0	41.02
Copper	0.627	20.10	34.04	34.04	0.0	40.97
Steel	0.627	19.63	34.04	34.04	0.0	42.32
PVC	0.627	21.18	34.04	34.04	0.0	37.77
Polyethylene	0.627	23.05	34.04	34.04	0.0	32.28

## DISCUSSION

An analytical investigation has been conducted in order to establish the applicability of similitude theory to cross-ply laminated plates. The results presented herein indicate that for elastic response of a cross-ply rectangular plate, based on structural similitude, a set of scaling laws can be found to develop design rules for small scale models. By establishing similarity conditions, the model parameters are specified. The accuracy of predicted prototype behavior by various models is investigated. The verification is based on the exact analytical solution of the model and its prototype. Theoretical compressive

and shear buckling loads of scale models are projected to predict corresponding prototype behavior. Even for models with different number of plies and stacking sequences (distorted models) the predicted data are well matched with full scale prototype data. For both load conditions the enforcement of the same response scaling law Eq.( 8) and Eq.( 14) yields models which yield accurate predictions. In the present study we assume that, except for isotropic models, prototype and models have the same aspect ratio ( $\lambda_R = 1$ ). This is not a necessary condition and in general ( $\lambda_R \neq 1$ ). The accuracy of prediction is very sensitive to the scale factor of the aspect ratios.

## CONCLUSIONS AND RECOMMENDATIONS

Small scale models can be used as a complement for analytical and computational investigations in solving the design problems of complex structures. This study indicates that a distorted model with a fewer number of layers can predict buckling load of the prototype with good accuracy.

Partial similarity based on direct use of governing equations is very convenient. There is tremendous freedom in design scale models because the number of similarity conditions is much smaller than the number of design variables.

Some recommendations for future research:

- Develop the method for designing and employing scale models for more complex systems, i.e stiffened and/or laminated curved configurations.
- Experimental verification of the accuracy of the purposed scaled model.
- Implementation of the structural similitude to inelastic and failure analysis of composite structures.

## ACKNOWLEDGMENTS

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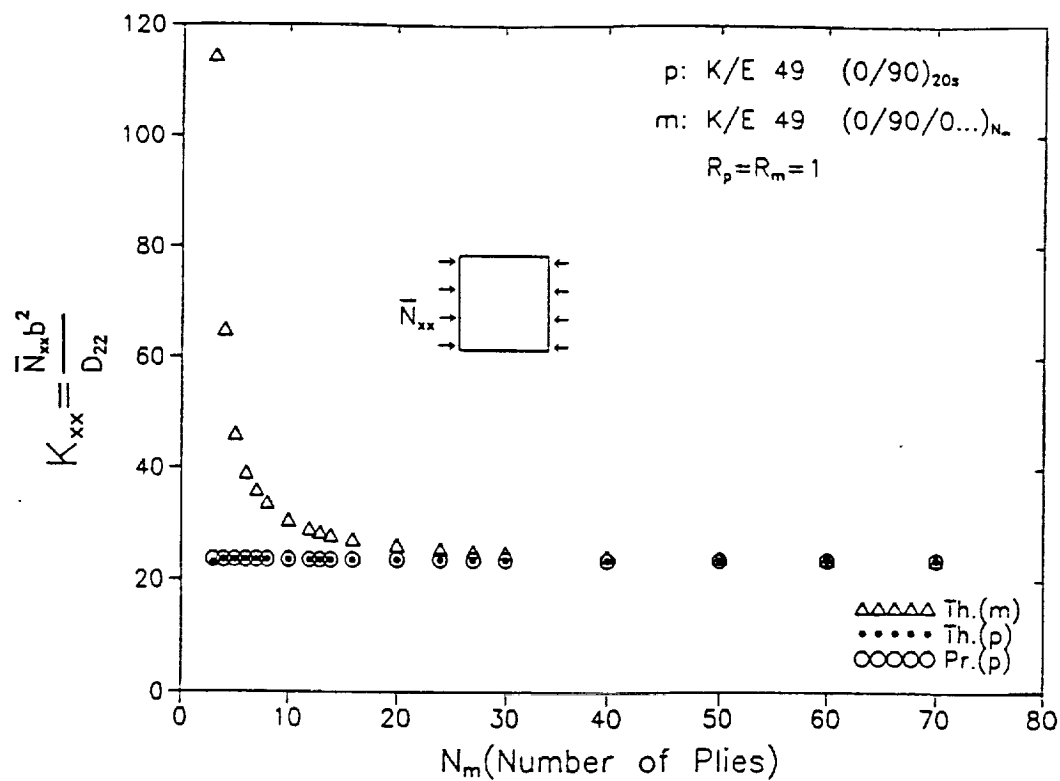


Figure 1: Predicted and theoretical compressive buckling load of the prototype (0/90)<sub>20s</sub> when (0/90/0...) <sub>n</sub> is used as model.

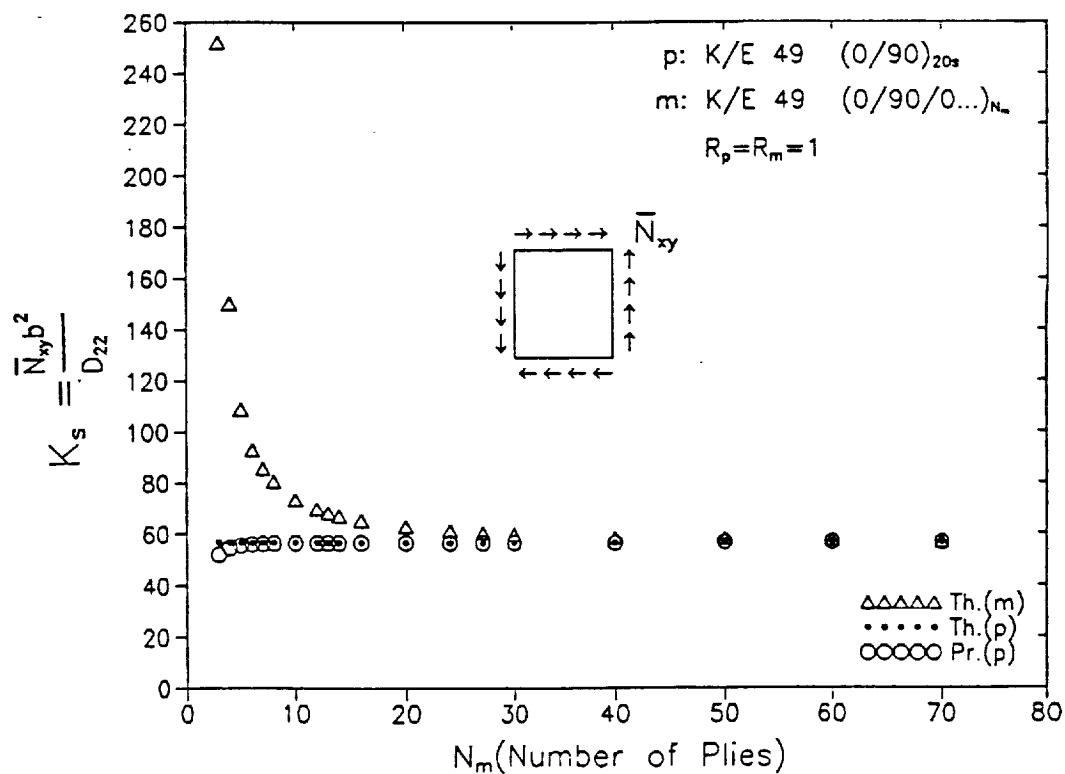


Figure 2: Predicted and theoretical shear buckling load of the prototype (0/90)<sub>20s</sub> when (0/90/0...) <sub>n</sub> is used as model.



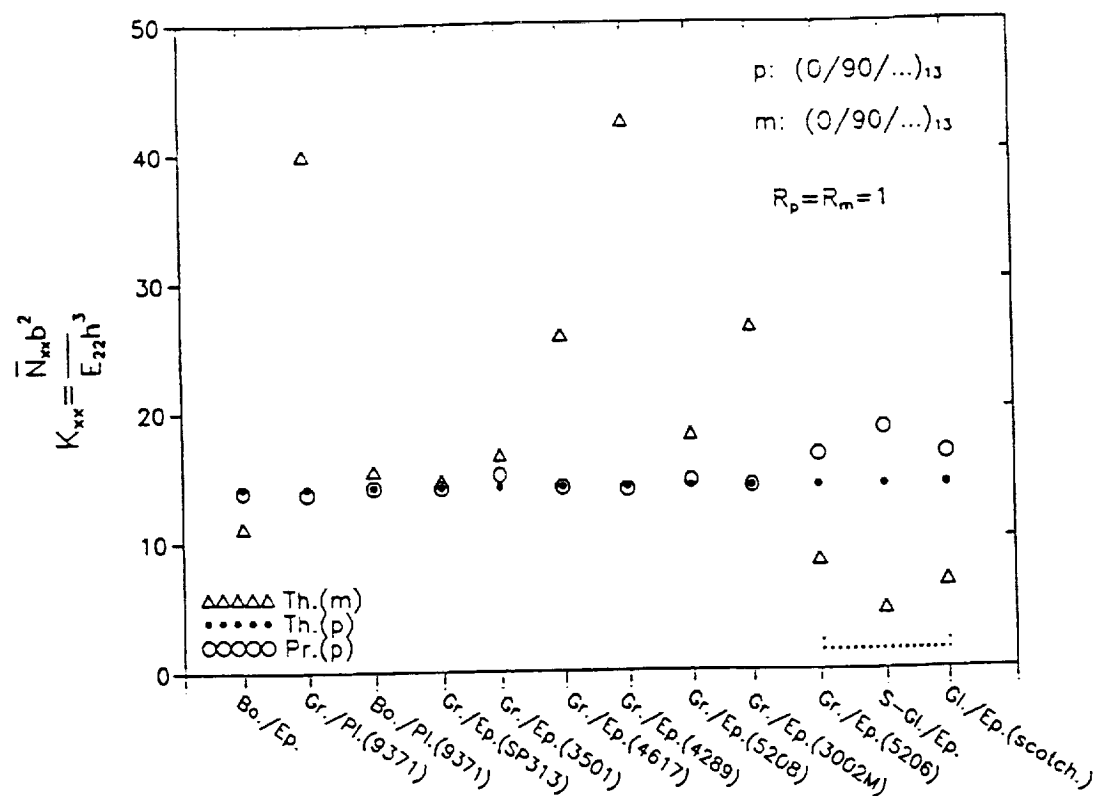


Figure 3: Predicted and theoretical compressive buckling load of the Kevlar/Epoxy prototype when model have different material properties.

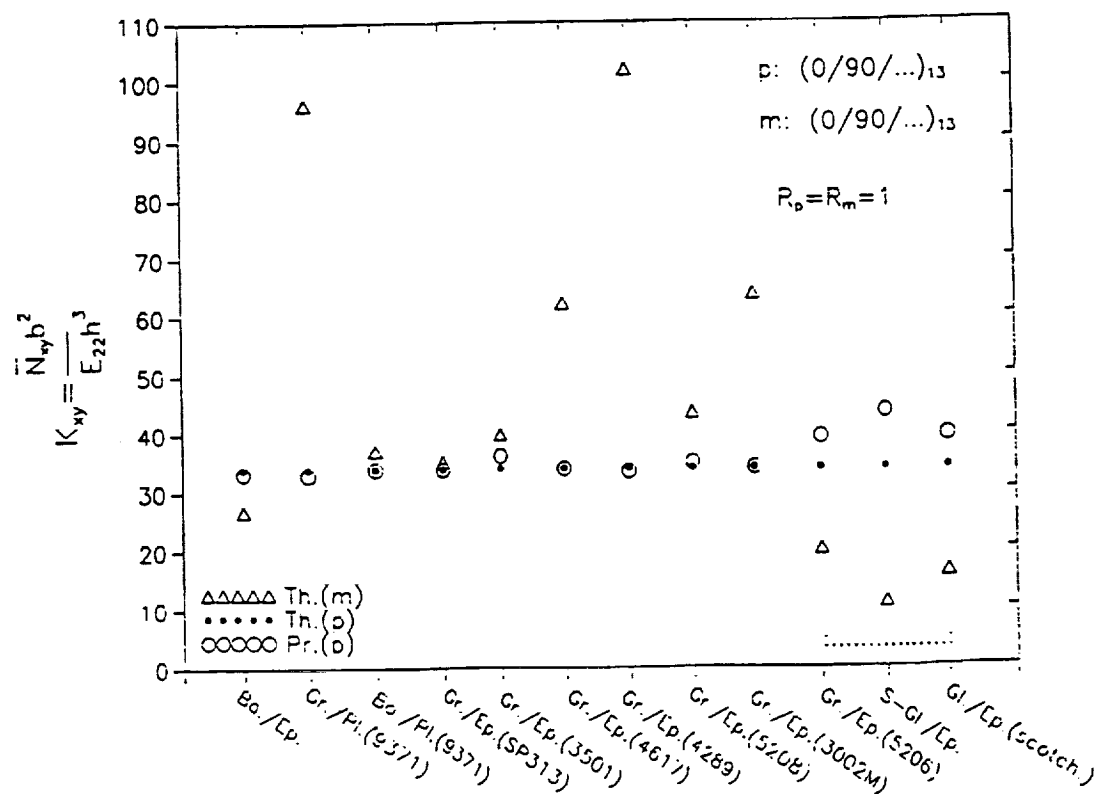


Figure 4: Predicted and theoretical shear buckling load of the Kevlar/Epoxy prototype when model have different material properties.

# DESIGN OF SCALED DOWN MODELS FOR STABILITY AND VIBRATION STUDIES

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## ABSTRACT

Use of reinforced composites in light-weight aerospace structures has increased steadily over the years. The outstanding mechanical and physical properties of advanced composites provide the engineer with potential to optimize properties specific to application. Since reinforced composite components require efficiency and wisdom in design, sophistication and accuracy in analysis, and numerous and careful experimental evaluations, there is a growing interest in small scale model testing<sup>1</sup>.

A scaled-down (by a large factor) model, *scale model*, which closely represents the structural behavior of the full-scale system, *prototype*, can prove to be an extremely beneficial tool. This possible development must be based on the existence of certain structural parameters that control the behavior of the structural system when acted upon by static and/or dynamic loads. If such structural parameters exist, a scaled-down replica can be built, which will duplicate the response of the full-scale system. The two systems are then said to be structurally similar. The term, then, that best describes this similarity is *Structural Similitude*.

Similitude theory is employed to develop the necessary similarity conditions (scaling laws). Scaling laws provide relationship between a full-scale structure and its scale models, and can be used to extrapolate the experimental data of a small, inexpensive, and testable model into design information for a large prototype. The difficulty of making completely similar scale models often leads to accept certain type of distortion from exact duplication of the prototype (partial similarity). Both complete and partial similarity are discussed. The procedure consists of systematically observing the effect of each parameter and corresponding scaling laws. Then acceptable intervals and limitations for these parameters and scaling laws are discussed. In each case, a set of valid scaling factors and corresponding response scaling laws that accurately predict the response of prototypes from experimental models is introduced.

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Particular emphasis is placed on the cases of free vibration and buckling of rectangular angle-ply laminated plates under uniaxial compressive and shear loads. This analytical study indicates that distorted models with a different number of layers, material properties, and geometries than those of the prototype can predict the behavior of the prototype with good accuracy.

The objectives of the investigation described herein are:

- create necessary similarity conditions in order to design a distorted model that accurately predicts prototype behavior.
  - distortion in stacking sequence and number of plies (N)
  - ply - level scaling and sublaminar - level scaling
  - distortion in material properties  $E_{ij}, \nu_{ij}, \rho$
  - distortion in fiber orientation angle  $\theta$
- evaluate the derived similarity conditions analytically.

In all of our work in this area we will restrict ourselves to linearly elastic material behavior. Furthermore, it is assumed that the laminates are free of damage (delaminations, matrix cracking, fiber breaks, etc.).

In this study, we consider only the procedure that is based on the direct use of the "governing equations". This method is more convenient than dimensional analysis, since the resulting similarity conditions are more specific and the relationships among variables are forced by the governing equations of the system.

Consider symmetric angle-ply laminated plates ( $B_{ij} = 0$ ). The plates are subjected to inplane normal and shear loads ( $\bar{N}_{xx}, \bar{N}_{yy}, \bar{N}_{xy}$ ). The buckling loads and vibration frequencies of symmetric angle-ply rectangular plates are described only by one differential equation<sup>2</sup>:

$$D_{11}w_{,xxxx}^0 + 4D_{16}w_{,xxyy}^0 + 2\bar{D}_{12}w_{,xyyy}^0 + 4D_{26}w_{,xyyy}^0 + D_{22}w_{,yyyy}^0 - \bar{N}_{xx}w_{,xx}^0 - \bar{N}_{yy}w_{,yy}^0 - \bar{N}_{xy}w_{,xy}^0 = \rho w_{,tt}^0 \quad (1)$$

where  $\bar{D}_{12} = D_{12} + 2D_{66}$

For a simply supported plate, the boundary conditions are

$$\text{at } x = 0, a \quad w = 0, \quad M_x = -D_{11}w_{,xx}^0 = 0$$

$$\text{at } y = 0, b \quad w = 0, \quad M_y = -D_{22}w_{,yy}^0 = 0$$

I. Shear Buckling: We now consider a simply supported plate which is subjected to the an in-plane shear stress ( $\bar{N}_{xx} = \bar{N}_{yy} = w_{,tt}^0 = 0$ ).

$$D_{11}w_{,xxxx}^0 + 4D_{16}w_{,xxxxy}^0 + 2\bar{D}_{12}w_{,xxxyy}^0 + 4D_{26}w_{,xyyy}^0 + D_{22}w_{,yyyy}^0 - \bar{N}_{xy}w_{,xy}^0 = 0 \quad (2)$$

The solution of the form  $w^0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b})$  satisfies all B.C.'s, but does not satisfy the buckling equation, Eq.( 2). Use of the Galerkin procedure, yields

$$\begin{aligned} (D_{11}\frac{m^4}{a^4} + 2\bar{D}_{12}\frac{m^2n^2}{a^2b^2} + D_{22}\frac{n^4}{b^4})A_{mn} = \\ \frac{32mn}{\pi^2ab} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \left( \frac{(m^2+p^2)}{a^2} D_{16} + \frac{(n^2+q^2)}{b^2} D_{26} + \frac{\bar{N}_{xy}}{4\pi^2} \right) A_{pq} Q_{mnpq} \end{aligned} \quad (3)$$

for  $m, n = 1, 2, \dots, \infty$ ; subject to the constraints  $m \pm p = \text{odd}$  and  $n \pm q = \text{odd}$ .

where  $R = \frac{a}{b}$ ,  $Q_{mnpq} = \frac{pq}{(m^2-p^2)(n^2-q^2)}$

$m, n, p$ , and  $q$  are integers which depend on the number of terms needed to approximate well the buckling mode shape(Symmetric/Anti-symmetric). By assuming the same aspect ratio for the model and its prototype ( $\lambda_R = 1$ ) and similar construction, model and prototype both can be well approximated by the same number of terms in the series with the same contribution of terms to the buckling mode and thus

$$\lambda_m = \lambda_n = \lambda_p = \lambda_q = 1 \quad \Rightarrow \quad \lambda_{\Omega} = 1, \lambda_{A_{mn}} = 1$$

Applying similitude theory to Eq.( 3) yields the following scaling laws

$$\lambda_{K_s} = \frac{\lambda_{D_{11}}}{\lambda_{E_{22}}\lambda_h^3\lambda_R^3} \quad (4)$$

$$\lambda_{K_s} = \frac{\lambda_{D_{12}}}{\lambda_{E_{22}}\lambda_h^3\lambda_R} \quad (5)$$

$$\lambda_{K_s} = \frac{\lambda_{D_{22}}}{\lambda_{E_{22}}\lambda_h^3}\lambda_R \quad (6)$$

$$\lambda_{K_s} = \frac{\lambda_{D_{16}}}{\lambda_{E_{22}}\lambda_h^3\lambda_R^2} \quad (7)$$

$$\lambda_{K_s} = \frac{\lambda_{D_{26}}}{\lambda_{E_{22}}\lambda_h^3} \quad (8)$$

$$\text{where } K_s = \frac{\bar{N}_{xy}b^2}{E_{22}h^3}$$

These conditions, Eqs.( 4)–( 8), contain both response and structural geometric parameters.

Complete Similarity : The necessary conditions for complete similarity between the model and its prototype is that all scaling laws be satisfied. This requirement yields

$$\lambda_{D_{11}} = \lambda_{D_{12}} = \lambda_{D_{22}} = \lambda_{D_{15}} = \lambda_{D_{26}} \quad (9)$$

By inspection it can be seen that these conditions, Eqs.( 9), are independent of ply thicknesses and they only depend on material properties and number of plies. So, two plates with different ply thickness but the same stacking sequences (i.e.(0/ + 45/90/ - 45)<sub>s</sub> and (0<sub>n</sub>/ + 45<sub>n</sub>/90<sub>n</sub>/ - 45<sub>n</sub>)<sub>s</sub>) satisfy Eqs.( 9). This is called ply-level scaling and it is the easiest way to achieved complete similarity. Table 1 presents the ply-level scaling for angle-ply plates under inplane shear loads.

So far, we have shown that in the special case of ply-level scaling similarity can be achieved. However, there are some constraints, in designing the model. These constraints involve the geometry of the model, the model material, the number of plies and the stacking sequence of laminates. Since this still appears to be restrictive, we allow the use of distortion in the design of the model.

Table 1 *Comparison of shear buckling loads of Graphite/Epoxy plates with ply - level scaling(complete similarity).*

model <i>Configuration</i>	$K_s = \frac{N_{xy}b^2}{E_2h^3}$			%Disc.	
	model	prototype	predicted	th.(p)&pr.(p)	th.(p)&th.(m)
(0/ + 45/90/ - 45) <sub>s</sub>	30.721	30.721	30.721	0.0	0.0
(0 <sub>3</sub> / + 45 <sub>3</sub> /90 <sub>3</sub> / - 45 <sub>3</sub> ) <sub>s</sub>	30.721	30.721	30.721	0.0	0.0
(0 <sub>10</sub> / + 45 <sub>10</sub> /90 <sub>10</sub> / - 45 <sub>10</sub> ) <sub>s</sub>	30.721	30.721	30.721	0.0	0.0

where  $\%Disc.(th.\&pr.) = 100 \times \frac{|theory - predicted|}{theory}$

Partial Similarity : Often complete similarity is difficult to achieve or even undesirable. When at least one of the similarity conditions can not be satisfied, partial similarity is achieved. In this case, the model which has some relaxation in similarity conditions is called a *distorted model*. These relaxations in the relationship between two systems cause model behavior to be different from that of the prototype. Since each variable has different influence on the response of the system, the resulting similarity conditions have different

influence. By understanding the effect of variables and similarity conditions over desired intervals, the similarity conditions which have the least influence can be neglected without introducing significant error<sup>3</sup>.

The choice of the "right" type of distortion is investigated as follows. In each case, all of the model parameters except one, are chosen to be identical to its prototype. Then, the effect of this relaxation for a wide range of this parameter is investigated.

Number of Plies : There are three ways to scale down the number of plies in a model. [a] ply-level scaling  $(0_n/+45_n/90_n/-45_n)_s$ , [b] sublamine level scaling<sup>4</sup>  $((0/+45/90/-45)_{n_s})$  and [c] general reduction of plies. The ply-level scaling leads to complete similarity (as already discussed). But the two other methods yield partial similarity. Figure 1 presents the shear buckling loads for models with different number of plies than those of the prototype and the predicted loads by using scaling laws depicted by Eq. (6).

Table 2 Accuracy of models with sublamine level scaling ( $N_p \neq N_m$ ) using non-dimensional load  $K_s$ ,  $\lambda_{E_s} = \lambda_{\nu_s} = 1$ ;  $(0/+45/90/-45)_{10_s}$ .

model	$K_s = \frac{N_{xy}b^2}{E_{22}h^3}$			%Disc.	
Configuration	model	prototype	predicted	th.(p)&pr.(p)	th.(p)&th.(m)
$(0/+45/90/-45)_{2_s}$	37.813	44.218	44.714	1.12	14.48
$(0/+45/90/-45)_{3_s}$	40.391	44.218	44.187	0.25	5.63

Material : Now we consider distortion in model material. For this purpose two different groups are considered: Isotropic materials (which include metals and plastics), and fiber reinforced composites. In all of these cases the prototype is an angle-ply laminated plate.

For the composite model, model and prototype have the same stacking sequence, number of plies  $((0/+45/90/-45)_{10_s})$  and aspect ratio. Figure 2 presents theoretical and predicted buckling loads of prototype and theoretical ones of the models for some typical composite materials. For the Kevlar/Epoxy prototype almost all considered materials can be used as the model material or vice versa.

Since plastics are used extensively in experimental studies of the behavior of the structures, the possibility of a plastic model or in general a model with isotropic material is considered. For isotropic materials, the assumption of  $\lambda_R = 1$  yields a model which cannot predict accurately the behavior of the prototype. By choosing  $R_m$  as a design parameter we are able to find isotropic models which yields excellent accuracy. Scaling law depicted

by Eq.( 6), yields acceptable aspect ratios for the models. Tables 3 present theoretical and predicted buckling loads for prototypes when the corresponding models are made of isotropic material.

Table 3 *Relaxation in material properties by using isotropic model,*  
*prototype is Kevlar/Epoxy (0/ + 45/90/ - 45)<sub>10s</sub> .*

model		$K_s = \frac{N_{xy} b^2}{E_m h^3}$			%Disc.	
material	$R_m$	model	prototype	predicted	th.(p)&pr.(p)	th.(p)&th.(m)
Aluminum	0.868	12.08	43.34	43.34	0.0	72.12
Copper	0.868	12.01	43.34	43.34	0.0	72.29
PVC	0.868	12.66	43.34	43.34	0.0	70.79

This study presents the applicability of small scale models, especially distorted models, in analyzing the elastic behavior of angle-ply laminated plates. Distorted models are more practical, since relaxation of each similarity condition eliminates some restrictions on the model design. The results presented herein indicate that, for elastic response of an angle-ply rectangular plate, based on structural similitude, a set of scaling laws can be found to develop design rules for small scale models. Results for buckling characteristics under uniaxial and vibration characteristic will be presented in the full paper.

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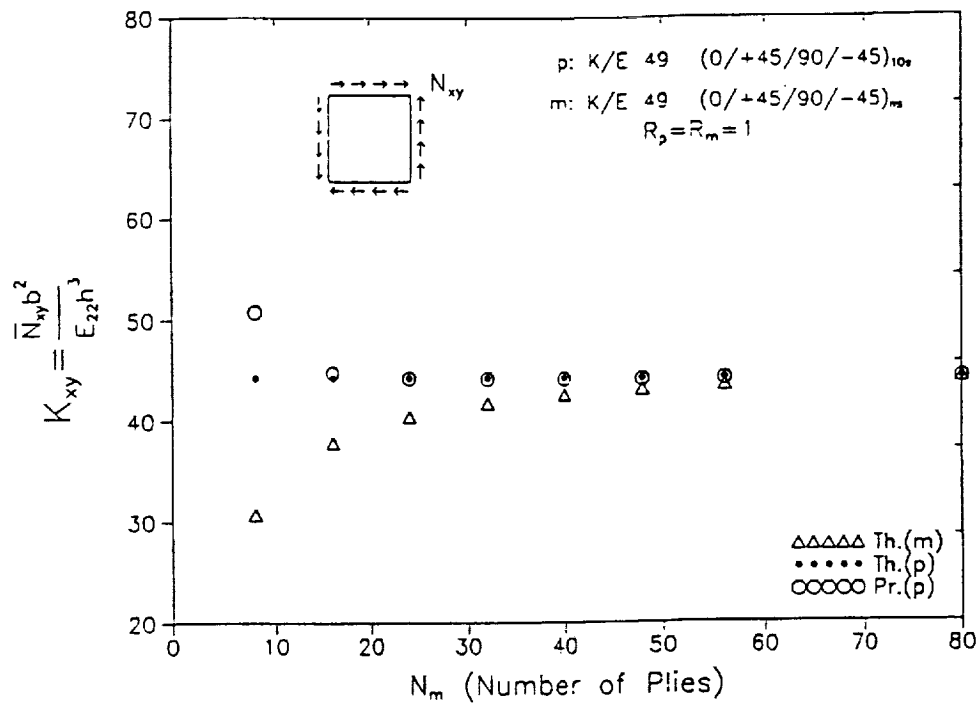


Figure 1: Predicted and theoretical shear buckling load of the prototype (0/+45/90/-45)<sub>10s</sub> when (0/+45/90/-45)<sub>m</sub> is used as model.

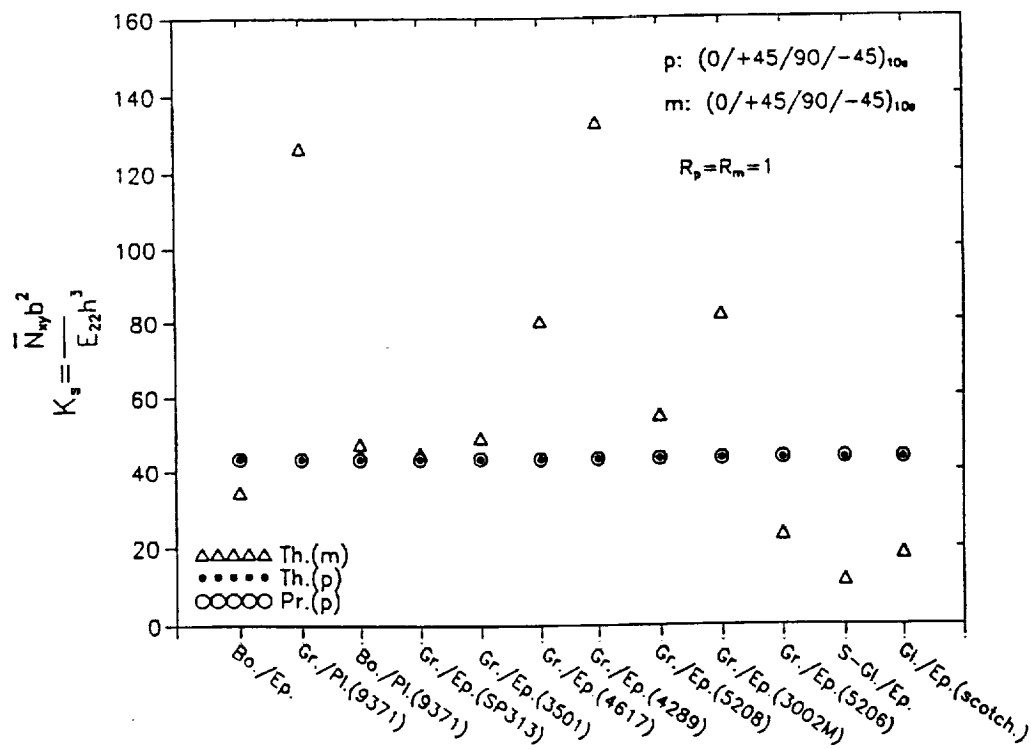


Figure 2: Predicted and theoretical shear buckling load of the Kevlar/Epoxy prototype when model have different material properties.



